

## Multiparticle Entanglement of Hot Trapped Ions

Klaus Mølmer\* and Anders Sørensen†

*Institute of Physics and Astronomy, University of Aarhus, DK-8000 Århus C, Denmark*

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We propose an efficient method to produce multiparticle entangled states of ions in an ion trap for which a wide range of interesting effects and applications have been suggested. Our preparation scheme exploits the collective vibrational motion of the ions, but it works in such a way that this motion need not be fully controlled in the experiment. The ions may, e.g., be in thermal motion and exchange mechanical energy with a surrounding heat bath without detrimental effects on the internal state preparation. Our scheme does not require access to the individual ions in the trap. [S0031-9007(99)08580-4]

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We describe in this Letter a method to construct states of the form

$$|\Psi\rangle = \frac{1}{\sqrt{2}} [e^{i\phi_g} |gg \cdots g\rangle + e^{i\phi_e} |ee \cdots e\rangle], \quad (1)$$

where  $|gg \cdots e\rangle$  and  $|ee \cdots e\rangle$  ( $N$  terms) are product states describing  $N$  ions which are all in the (same) internal state  $g$  or  $e$ . The unique correlation between different particles in such an entangled state is a significant nonclassical feature of quantum mechanics. Entangled states with two particles have been employed to test Bell's inequality and to rule out local realistic descriptions of nature [1]. States like (1) with three particles, the so-called Greenberger-Horne-Zeilinger (GHZ) states [2], and with more particles allow a closer scrutiny of the role of quantum correlations, and there is a large current interest in carrying out experiments on such states. It has also been shown that the states (1) are ideal for spectroscopic investigations [3], and the study of entangled states of many particles is in line with the rapidly growing interest in quantum encoding and quantum information processing [4]. Together with an analysis of their applicability within high precision frequency measurements, a method to produce such states in an ion trap was proposed in [3]. That method requires experimental control of two collective vibrational modes of the ions, which are involved in stepwise manipulations of the internal states of the ions.

In order to describe our more straightforward proposal, we shall show that the desired state (1) can be obtained by application of one simple interaction Hamiltonian. We shall then show that this Hamiltonian is realized in an ion trap by simply illuminating all the ions with laser beams with two different colors, and we shall present numerical results that display the performance of our proposal under quite relaxed assumptions for the functioning of the ion trap.

We begin with all ions in the ground state  $|g\rangle$ , and we assume that our external perturbation interacts in the same way with all ions, so that the joint internal state remains symmetric with respect to interchange of the ions.

In such a situation, it is convenient to apply the spin representation of atomic states, where a collective state of all ions is represented by  $|J, M\rangle$  which is an eigenstate of the  $J_z$  operator,  $J_z|J, M\rangle = \hbar M|J, M\rangle$ ,  $M = -J, \dots, J$ , where  $\hbar$  is Planck's constant.  $J_z$  "counts" the number of excited ions, so that a state, invariant under permutation of the ions, with a certain number  $N_e$  of excited ions is described as  $|J = N/2, M = N_e - N/2\rangle$ , and the state (1) is a superposition of the states  $|N/2, -N/2\rangle$  and  $|N/2, N/2\rangle$ . An increase of the quantum number  $M$  by unity corresponds to a single ionic excitation, and it is described by the angular momentum ladder operator  $J_+|J, M\rangle = \hbar\sqrt{(J+M+1)(J-M)}|J, M+1\rangle$ , where the normalization factor accounts precisely for the combinatorial aspects of this excitation of the system. As we shall detail below, it is possible to apply classical laser irradiation which cannot be absorbed by a single ion, but which requires two ions to undergo the  $|g\rangle \leftrightarrow |e\rangle$  transition, simultaneously. When this field is applied to all ions in the trap, the effective Hamiltonian describing the evolution of the state of the ions in the spin representation can be written,

$$H = \frac{\chi}{\hbar} (J_+^2 + J_-^2 + J_+J_- + J_-J_+) = 4 \frac{\chi}{\hbar} J_x^2, \quad (2)$$

where  $J_-$  is the Hermitian conjugate of  $J_+$  and where  $J_x = (J_+ + J_-)/2$  denotes the  $x$  component of the effective spin.

Time evolution is accommodated by application of the operator  $U(t) = \exp(-i4\chi J_x^2 t/\hbar^2)$  to the initial state  $|N/2, -N/2\rangle$  and the population becomes distributed on all  $|J, M\rangle$  states, with  $M$  differing from  $-N/2$  by an even number. If  $N$  is even, population is transferred all the way to the state  $|N/2, N/2\rangle$ , which is the second component of (1). Moreover, at the instant  $t = \pi/8\chi$ , the state is precisely of the form (1), with  $\phi_g = -\pi/4$  and  $\phi_e = \pi/4 + N\pi/2$ . We show in Fig. 1 the time evolution of the populations of the two extremal states for different (even) numbers of ions. At  $t = \pi/8\chi$  and at later odd multiples of this value, both populations equal one half.

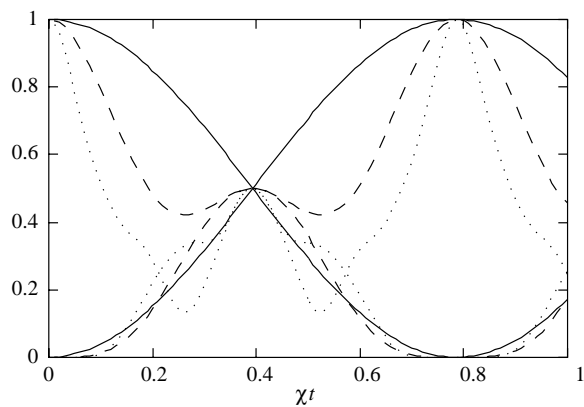


FIG. 1. Time evolution of the population of the joint ionic ground state  $|gg \cdots g\rangle = |N/2, -N/2\rangle$  (curves starting from the value of unity at  $t = 0$ ), and the joint ionic excited state  $|ee \cdots e\rangle = |N/2, N/2\rangle$ . Results are presented for different values of the number of ions:  $N = 2$  (solid curves),  $N = 4$  (dashed curves) and  $N = 8$  (dotted curves). At  $t = \pi/8\chi$  the states are in a 50-50 superposition and the state (1) is obtained.

This result can be understood from the rotation properties of angular momenta: The initial state can be expanded on eigenstates of  $J_x$ :  $|N/2, -N/2\rangle = \sum_M c_M |N/2, M\rangle_x$ , and from the properties of the Wigner rotation functions [5], it follows that  $|N/2, N/2\rangle = \sum_M c_M (-1)^{N/2-M} |N/2, M\rangle_x$ . Inserting the values for  $\phi_g$  and  $\phi_e$  mentioned above, we can therefore write the state (1) as  $\sum_M c_M \frac{1}{\sqrt{2}} (e^{-i\pi/4} + (-1)^M e^{i\pi/4}) |N/2, M\rangle_x$ . The net factors multiplying the initial amplitudes  $c_M$  are unity for  $M$  even and  $-i$  for  $M$  odd. The action of  $U(t)$  in the  $J_x$  basis amounts to a multiplication of each amplitude  $c_M$  by  $\exp(-4i\chi M^2 t)$ , and for  $t = \pi/(8\chi)$  this factor just attains the desired value of unity for  $M$  even and  $-i$  for  $M$  odd.

Note that the state (1) is not only a multiparticle entangled state; it is a superposition of two “mesoscopically distinguishable” states. In quantum optics, such states have been studied experimentally for single quantum systems, such as a harmonically trapped ion [6] and a single mode of the quantized radiation field [7], excited into superpositions of states of “mesoscopic” separation. Mathematically, our analysis above resembles the one for production of such superpositions by propagation of an optical field through an amplitude dispersive medium [8].

We now turn to a “microscopic” analysis of a collection of  $N$  two-level ions in a linear ion trap. The choice of a linear trap eliminates the so-called micromotion for ions on the axis, and by choosing the transverse confinement much stronger than the longitudinal one, we can neglect coupling to transverse vibrations. Because of the Coulomb repulsion among the ions, the various longitudinal vibrational modes have different frequencies, and by choosing the laser detuning close to the center-of-mass vibrational frequency we may exclude other modes from our analysis [9].

The quantized energy of the total system is composed of the center-of-mass vibrational energy of the ion string and the internal electronic energy of the ions. The vibrational energy is represented by a harmonic oscillator with ladder operators  $a$  and  $a^\dagger$  and frequency  $\nu$ , and the internal energy is described by Pauli matrices  $\sigma_{zi}$  and the energy difference  $\hbar\omega_{eg}$ ,

$$H_0 = \hbar\nu(a^\dagger a + 1/2) + \frac{\hbar\omega_{eg}}{2} \sum_i \sigma_{zi}. \quad (3)$$

The ions interact with two laser fields with frequencies  $\omega_j$  ( $j = 1, 2$ ). We assume that both fields have the same intensity and we also assume that the ion-laser interaction strength characterized by the Rabi frequency  $\Omega$  is the same for all ions. Pauli matrices  $\sigma_{\pm i}$  describe internal atomic transitions and the interaction Hamiltonian can be written

$$H_{\text{int}} = \sum_{i,j} \frac{\hbar\Omega}{2} (\sigma_{+i} e^{i[\eta_j(a+a^\dagger) - \omega_j t]} + \text{H.c.}). \quad (4)$$

[H.c. in (4) denotes the Hermitian conjugate.] The Dicke parameters  $\eta_j$  are products of the projection of the  $k$  vector of the  $j$ th laser field along the direction of the string of ions and the rms excursion of the ionic center-of-mass along this direction [10]. In practical realizations the  $k$  vectors can be nearly identical for the two fields, and we assume  $\eta_1 = \eta_2 = \eta$ . Numerical solutions of the Schrödinger equation with the Hamiltonian given by Eqs. (3) and (4), are readily obtained within the basis of  $2^N$  internal states and a sufficiently large set of vibrational states  $\{|f_1 f_2 \cdots f_N\rangle |n\rangle\}$ ,  $f_i = e$  or  $g$ . First, however, we discuss a perturbative analysis pointing to appropriate choices of parameters which will effectively yield the spin Hamiltonian (2).

The energy levels of the ion string constitute a ladder of equidistant vibrational levels added to the internal ionic energies. This ladder structure has been extensively used in laser manipulation of ions in traps, where laser frequencies differing from the atomic transition frequency by a multiple of the trap frequency may be used to control the internal and external dynamics [6], and excitation of collective vibrational modes for several ions has been proposed as a means for implementing quantum logic in an ion trap quantum computer [11].

Here, we make use of the idea we put forward in [12] to apply laser fields with two different frequencies so that the two-photon process, exciting any pair of ions in the trap  $|gg\rangle \rightarrow |ee\rangle$  is resonant, i.e.,  $\omega_1 + \omega_2 = 2\omega_{eg}$ , but neither of the frequencies are resonant with single excitations of the ions. By choosing  $\omega_1 = \omega_{eg} - \delta$  and  $\omega_2 = \omega_{eg} + \delta$ , where  $\delta$  is close to, but not resonant with, the center-of-mass vibration frequency we may limit our perturbative analysis to involve only the intermediate states with one excited ion and a vibrational quantum number raised or lowered by unity. Note that the lack of degeneracy of vibrational modes is crucial because

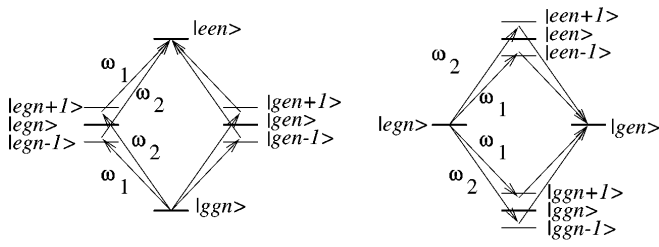


FIG. 2. Level scheme for a pair of ions sharing an oscillator degree of freedom. Left: By application of laser light with frequencies  $\omega_{eg} \pm \delta$ , where  $\delta$  is somewhat smaller than the vibrational frequency  $\nu$ , we identify four transition paths between the states  $|gg\rangle|n\rangle$  and  $|ee\rangle|n\rangle$ , which interfere as described in the text. Right: Four similar transition paths are identified between states  $|eg\rangle|n\rangle$  and  $|ge\rangle|n\rangle$ , yielding the same effective coupling among these states as between the states in the left panel.

it prevents a number of modes to interfere and obstruct the collective transitions in the multiparticle system. We show in Fig. 2 the effective transition paths between states  $|gg\rangle|n\rangle$  and  $|ee\rangle|n\rangle$  and between  $|eg\rangle|n\rangle$  and  $|ge\rangle|n\rangle$  mediated by the interaction with the laser fields within any pair of ions. Although they serve as intermediate states in the process, the intermediate states are only virtually i.e., negligibly, populated. The interaction strengths of the different one-photon transitions in Fig. 2 depend on the value of the vibrational quantum number  $n$  and in the limit of  $\eta\sqrt{n+1} \ll 1$ , terms between states  $n$  and  $n+1$  are proportional to  $\sqrt{n+1}$  whereas terms between  $n-1$  and  $n$  are proportional to  $\sqrt{n}$ . When we add the contributions from the four different transition paths shown in either panel of Fig. 2, a “miracle” occurs: The paths (virtually) involving intermediate states with  $n+1$  vibration quanta yield  $(\sqrt{n+1}\Omega\eta)^2/(\delta-\nu)$  and the ones with  $n-1$  quanta yield  $(\sqrt{n}\Omega\eta)^2/(\nu-\delta)$ . The signs of the denominators cause a destructive interference which exactly removes the  $n$  dependence from the total effective Rabi frequency of the two-photon transition. This implies that the internal state evolves in a manner independent of the external state, which may be in a mixed (e.g., thermal) state and which may even evolve with time.

When the summation over all ions is carried out, the couplings lead to the spin Hamiltonian (2) with

$$\chi = \frac{\eta^2\Omega^2\nu}{2(\nu^2 - \delta^2)}, \quad (5)$$

and to the extent that our perturbative treatment is valid, we therefore expect to realize the results, presented in Fig. 1. In Fig. 3, we show numerical simulations of the dynamics governed by the correct Hamiltonian (3),(4) for 4 ions, i.e., beyond perturbation theory and to all orders in the Dicke parameter  $\eta$ . We assume a distribution on vibrational levels with quantum numbers ranging between 0 and 40, and we assume that the center-of-mass motion exchanges energy with a surrounding thermal reservoir rep-

resented by relaxation operators  $c_1 = \sqrt{\Gamma(n_{th} + 1)}a$  and  $c_2 = \sqrt{\Gamma n_{th}}a^\dagger$ . Our calculation for a reservoir with a mean excitation of  $n_{th} = 5$  vibrational quanta at the trap frequency and a coupling strength  $\Gamma = 0.0001$  may represent a heating mechanism towards higher temperatures counteracted by laser cooling on one ion in the string, reserved for this purpose. The action of the reservoir is described by quantum jumps and a non-Hermitian Hamiltonian in a Monte Carlo wave function simulation [13]. Figure 3(a) represents the result of a single Monte Carlo realization with quantum jumps in the vibrational state at instants of time, indicated by arrows Fig. 3(a). Figure 3(b) is an average over ten such realizations. The full lines present the populations and coherences of the

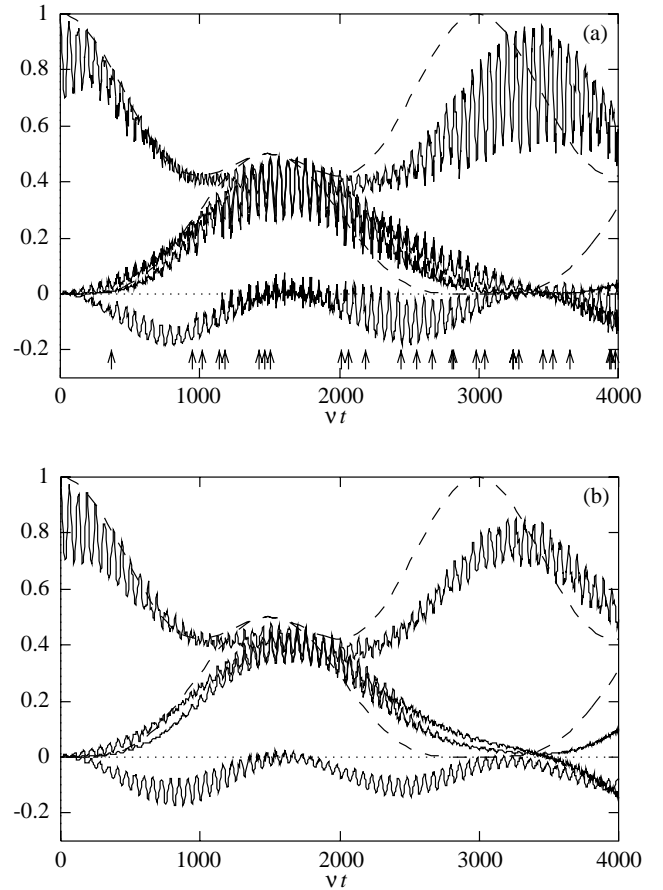


FIG. 3. Results of numerical integration of the Schrödinger equation for four ions. The physical parameters are  $\delta = 0.9\nu$ ,  $\Omega = 0.1\nu$ ,  $\eta = 0.1$ ,  $\Gamma = 0.0001\nu$ , and  $n_{th} = 5$ . (a) Single Monte Carlo realization with quantum jumps in the vibrational states at instants of time, indicated by arrows in the figure. (b) Average over ten realizations. The first full curve (counted from above at  $\nu t < 1000$ ) is the population of the joint ground state, the second one is the imaginary part of the internal density matrix element between the joint ground and the joint excited state, the third one is the population of the joint excited state, and the fourth one is the real part of the off-diagonal density matrix element. The dashed curves are equivalent to the  $N = 4$  curves of Fig. 1, obtained by application of the spin Hamiltonian (2) with  $\chi = \frac{\eta^2\Omega^2\nu}{2(\nu^2 - \delta^2)}$ .

internal state of the ions, obtained by tracing out the vibrational degrees of freedom. For comparison we also show the result of a calculation with the corresponding spin Hamiltonian (2) (dashed curve). With the realistic parameters chosen, the state at  $\nu t \sim 1500$  is very close to the desired state in (1). Small oscillatory deviations from the smooth results of Fig. 1 are seen. The magnitude of the oscillations is given by the physical parameters, so that in the limit of small  $\eta$  and  $\Omega$ , they can be made arbitrarily small [12].

The pairwise interaction with ions does not produce a coherent coupling of the two components in (1) if  $N$  is odd. In this case, however, one can verify within the spin model, that it suffices to apply a linear coupling  $H_1 = 4\xi J_x$  for the duration  $t = \pi/8\xi$  in addition to application of our quadratic term (2). Since  $J_x$  and  $J_x^2$  commute, the linear Hamiltonian may actually be applied before, after, or simultaneously with the quadratic one. This additional operation is easy to implement in the ion trap, where one only needs to drive the ions on the atomic resonance frequency  $\omega_{eg}$ .

In summary, we have presented a method to prepare a multiparticle entangled state of trapped ions. Our suggestion aims at experimental application, and it is relevant for experiments with limited control of the vibrational motion, where the ions heat up for different technical reasons, or where there is not access to individual ions. The possibility to exercise control over some quantum degrees of freedom of a system while others are uncontrolled is commonplace in coherent optical excitation experiments, where the internal states of atoms in a gas may be controlled regardless of the external thermal motion. Our analysis adds the surprising possibility of coherently controlling the mutual internal state of several particles by explicit use of the uncontrolled, incoherently excited degrees of freedom. We believe that much wider possibil-

ities for quantum state control become feasible with the introduction of the crucial elements of our method: resonance conditions allowing only pairwise transitions, virtual excitations of uncontrolled degrees of freedom, and partially destructive interference between transition paths. We have recently applied these ideas to quantum computation in an ion trap [12] and we are currently investigating their possible use in other physical systems.

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\*Email address: moelmer@ifa.au.dk

†Email address: anderss@ifa.au.dk

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