

Mathematical formulae

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Chapter 1

Some basics

1.1 Kronecker and Levi-Civita symbols

$$\delta_{ij} = 1 \text{ if } i = j, 0 \text{ otherwise} \quad (1.1)$$

$$\begin{aligned} \epsilon_{ijk} &= 1 \text{ if } ijk \text{ are an even permutation of } 123 \\ &= -1 \text{ if } ijk \text{ are an odd permutation of } 123 \\ &= 0 \text{ if } i = j \text{ or } i = k \text{ or } j = k \end{aligned}$$

$$\sum_{i,j} \epsilon_{ijk} \epsilon_{ijn} = 2\delta_{kn} \quad (1.2)$$

$$\sum_i \epsilon_{ijk} \epsilon_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km} \quad (1.3)$$

1.2 Taylor series

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots + \frac{n!}{(n-r)!r!}x^r + \dots \quad (1.4)$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad (1.5)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad (1.6)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad (1.7)$$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \dots \quad (1.8)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad (1.9)$$

1.3 Trigonometric relations

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \quad (1.10a)$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \quad (1.10b)$$

$$2 \sin A \sin B = \cos(A - B) - \cos(A + B) \quad (1.11a)$$

$$2 \cos A \cos B = \cos(A + B) + \cos(A - B) \quad (1.11b)$$

$$2 \sin A \cos B = \sin(A + B) + \sin(A - B) \quad (1.11c)$$

$$\sin 2A = 2 \sin A \cos A \quad (1.12a)$$

$$\cos 2A = \cos^2 A - \sin^2 A \quad (1.12b)$$

$$\sin \frac{1}{2}A = \sqrt{\frac{1}{2}(1 - \cos A)} \quad (1.12c)$$

$$\cos \frac{1}{2}A = \sqrt{\frac{1}{2}(1 + \cos A)} \quad (1.12d)$$

$$\sin^2 A = \frac{1}{2}(1 - \cos 2A) \quad (1.12e)$$

$$\cos^2 A = \frac{1}{2}(1 + \cos 2A) \quad (1.12f)$$

1.4 Hyperbolic trigonometric relations

$$\cosh^2 x - \sinh^2 x = 1 \quad (1.13a)$$

$$\tanh^2 x + \operatorname{sech}^2 x = 1 \quad (1.13b)$$

$$\operatorname{ctnh}^2 x - \operatorname{csch}^2 x = 1 \quad (1.13c)$$

$$(1.13d)$$

$$\sinh(ix) = i \sin x \quad (1.14a)$$

$$\cosh(ix) = \cos x \quad (1.14b)$$

$$(1.14c)$$

1.5 Logarithms

The logarithm to base a is defined by

$$x = a^{\log_a x}$$

for any nonnegative x .

$$\begin{aligned}\log_a(1) &= 0 \\ \log_a(xy) &= \log_a(x) + \log_a(y) \\ \log_a(x/y) &= \log_a(x) - \log_a(y) \\ \log_a(x^p) &= p \log_a(x)\end{aligned}$$

The above relations hold for any base.

Taking $a = e$ we see that

$$x = e^{\log_e x} = e^{\ln x}$$

where $\ln = \log_e$ is the natural logarithm, i.e. logarithm to base e .

So

$$e^{\ln x} = a^{\log_a x} = e^{\ln a \log_a x}$$

and

$$\log_a x = \frac{\ln x}{\ln a}.$$

More generally

$$\log_b x = \frac{\log_a x}{\log_a b},$$

where a is an arbitrary base.

1.6 Solution of cubic equations

Write the cubic equation as

$$z^3 + a_2 z^2 + a_1 z + a_0 = 0. \quad (1.15)$$

Define

$$q = \frac{a_1}{3} - \frac{a_2^2}{9} \quad (1.16)$$

$$r = \frac{a_1 a_2 - 3a_0}{6} - \frac{a_2^3}{27} \quad (1.17)$$

$$h = q^3 + r^2 \quad (1.18)$$

For $h > 0$ there are one real root and a pair of complex conjugate roots, for $h = 0$ all roots are real, and at least two are equal, and for $h < 0$ all roots are real.

The three roots are

$$z_1 = s_1 + s_2 - \frac{a_2}{3}, \quad (1.19)$$

$$z_2 = -\frac{s_1 + s_2}{2} - \frac{a_2}{3} + i \frac{\sqrt{3}}{2} (s_1 - s_2), \quad (1.20)$$

$$z_3 = -\frac{s_1 + s_2}{2} - \frac{a_2}{3} - i \frac{\sqrt{3}}{2} (s_1 - s_2), \quad (1.21)$$

where

$$s_1 = (r + \sqrt{h})^{1/3}, \quad s_2 = (r - \sqrt{h})^{1/3}. \quad (1.22)$$

Note

$$z_1 + z_2 + z_3 = -a_2, \quad (1.23)$$

$$z_1 z_2 + z_1 z_3 + z_2 z_3 = a_1, \quad (1.24)$$

$$z_1 z_2 z_3 = -a_0. \quad (1.25)$$

1.7 Combinatorial factors

Binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad n, k \text{ integers} \quad (1.26)$$

Binomial series

$$(1+x)^a = \sum_{k=0}^{\infty} \binom{a}{k} x^k = 1 + ax + \frac{a(a-1)}{2!} x^2 + \dots \quad (1.27)$$

Here a is an arbitrary complex number and $\binom{a}{k} \equiv \frac{a(a-1)(a-2)\dots(a-k+1)}{k!}$.

The binomial distribution $B_p(n, N)$ gives the probability of obtaining n successes in N trials when a single trial yields success with probability p .

$$B_p(n, N) = \binom{N}{n} p^n (1-p)^{N-n}. \quad (1.28)$$

1.8 Matrices

The Kronecker product of a matrix \mathbf{A} with dimensions $m \times n$ and a matrix \mathbf{B} with dimensions $p \times q$ is a matrix $\mathbf{C} = \mathbf{A} \otimes \mathbf{B}$ with dimensions $mp \times nq$. The elements of the matrices are related by

$$\mathbf{C}_{\alpha\beta} = \mathbf{A}_{ij} \mathbf{B}_{kl}$$

where

$$\alpha = p(i-1) + k,$$

$$\beta = q(j-1) + l.$$

Given α, β we have the inverse relations

$$i = \text{int}(\alpha/p) + 1,$$

$$j = \text{int}(\beta/q) + 1,$$

$$k = \alpha - p(i-1),$$

$$l = \beta - q(j-1).$$

Chapter 2

Poisson distribution

The Poisson distribution is

$$P(n, \bar{n}) = \frac{\bar{n}^n e^{-\bar{n}}}{n!}.$$

This gives the probability of observing n events when the mean number observed is \bar{n} . For large \bar{n} the Poisson distribution becomes Gaussian. To see this assume $x = n$ is a real variable, consider $\ln P$ and use the Stirling formula $n! \simeq \sqrt{2\pi n} n^n e^{-n}$ to get

$$\ln P = x \ln(\bar{n}/x) + (x - \bar{n}) - \ln \sqrt{2\pi x}.$$

Then put $y = x - \bar{n}$, assume $y \ll \bar{n}$, and use $\ln \sqrt{2\pi x} \simeq \ln \sqrt{2\pi \bar{n}}$ to get

$$\ln P = -\frac{y^2}{2\bar{n}} - \ln \sqrt{2\pi \bar{n}}.$$

Therefore

$$P = \frac{1}{\sqrt{2\pi \bar{n}}} e^{-y^2/2\bar{n}} = \frac{1}{\sqrt{2\pi \bar{n}}} e^{-(x-\bar{n})^2/2\bar{n}}$$

which is a Gaussian with mean \bar{n} and standard deviation $\sigma_n = \sqrt{\bar{n}}$.

The *FWHM* of the distribution is found from

$$e^{-(\bar{n} + FWHM/2 - \bar{n})^2/2\bar{n}} = \frac{1}{2}$$

so

$$FWHM = (8 \ln(2) \bar{n})^{1/2} \simeq 2.355 \sqrt{\bar{n}}.$$

Chapter 3

Curvilinear coordinates

3.1 3D Cartesian (x, y, z)

$$\mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\text{line element } ds = dx\hat{x} + dy\hat{y} + dz\hat{z}$$

$$\text{volume element } d\mathbf{r} = d^3r = dxdydz$$

$$\nabla\psi = \hat{x}\frac{\partial\psi}{\partial x} + \hat{y}\frac{\partial\psi}{\partial y} + \hat{z}\frac{\partial\psi}{\partial z} \quad (3.1a)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} \quad (3.1b)$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad (3.1c)$$

$$\nabla \times \mathbf{A} = \hat{x} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{y} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{z} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \quad (3.1d)$$

3.2 3D cylindrical (ρ, ϕ, z)

$$\text{line element } ds = d\rho\hat{\rho} + \rho d\phi\hat{\phi} + dz\hat{z}$$

$$\text{volume element } d\mathbf{r} = d^3r = \rho d\rho d\phi dz$$

$$\rho = \sqrt{x^2 + y^2}, \quad \phi = \tan^{-1} \left(\frac{y}{x} \right), \quad z = z \quad (3.2a)$$

$$x = \rho \cos \phi, \quad y = \rho \sin \phi, \quad z = z \quad (3.2b)$$

$$\nabla\psi = \hat{\rho}\frac{\partial\psi}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \hat{z}\frac{\partial\psi}{\partial z} \quad (3.3a)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} \quad (3.3b)$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z} \quad (3.3c)$$

$$\nabla \times \mathbf{A} = \hat{\rho}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right) + \hat{z}\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial\phi}\right) \quad (3.3d)$$

3.3 3D spherical (r, θ, ϕ)

line element $ds = dr\hat{r} + r d\phi\hat{\phi} + r \sin\theta d\theta\hat{\theta}$

volume element $dr = d^3r = r^2 \sin\theta dr d\theta d\phi$

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \cos^{-1}\left(\frac{z}{r}\right), \quad \phi = \tan^{-1}\left(\frac{y}{x}\right) \quad (3.4a)$$

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r \cos\theta \quad (3.4b)$$

$$\nabla\psi = \hat{r}\frac{\partial\psi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{\phi}\frac{1}{r \sin\theta}\frac{\partial\psi}{\partial\phi} \quad (3.5a)$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} \quad (3.5b)$$

$$\nabla \cdot \mathbf{A} = \left(\frac{2}{r} + \frac{\partial}{\partial r}\right)A_r + \frac{1}{r \sin\theta}\frac{\partial A_\phi}{\partial\phi} + \left(\frac{1}{r}\frac{\partial}{\partial\theta} + \frac{\cot\theta}{r}\right)A_\theta \quad (3.5c)$$

$$\nabla \times \mathbf{A} = \hat{r}\frac{1}{r \sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right) + \hat{\theta}\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(r A_\phi)\right) + \hat{\phi}\frac{1}{r}\left(\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial\theta}\right) \quad (3.5d)$$

3.4 Vector identities

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \quad (3.6a)$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \quad (3.6b)$$

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}) \quad (3.6c)$$

$$(\mathbf{A} \times \mathbf{B})_i = \sum_{j,k} \epsilon_{ijk} A_j B_k \quad (3.7)$$

$$\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = \sum_{i,j,k} \epsilon_{ijk} C_i A_j B_k \quad (3.8)$$

Derivatives of products

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) \quad (3.9a)$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B} \quad (3.9b)$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \quad (3.9c)$$

$$\nabla \cdot (\psi \mathbf{A}) = \mathbf{A} \cdot \nabla \psi + \psi \nabla \cdot \mathbf{A} \quad (3.10a)$$

$$\nabla \times (\psi \mathbf{A}) = \nabla \psi \times \mathbf{A} + \psi \nabla \times \mathbf{A} \quad (3.10b)$$

Second derivatives

$$\nabla \times \nabla \psi = 0 \quad (3.11a)$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \quad (3.11b)$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \quad (3.11c)$$

$$\nabla \cdot \mathbf{r} = 3, \quad \nabla \times \mathbf{r} = 0 \quad (3.12a)$$

$$\nabla \cdot \hat{\mathbf{r}} = \frac{2}{r}, \quad \nabla \times \hat{\mathbf{r}} = 0 \quad (3.12b)$$

For any constant vector \mathbf{A} ,

$$\mathbf{A} = \nabla \times \left(\frac{\mathbf{A} \times \mathbf{r}}{2} \right)$$

Chapter 4

Complex numbers

Define $z = x + iy$. Then

$$z^* = x - iy \quad (4.1a)$$

$$|z|^2 = zz^* = x^2 + y^2 \quad (4.1b)$$

$$|z| = \sqrt{x^2 + y^2} \quad (4.1c)$$

$$\arg(z) = \tan^{-1}(y/x) \quad (4.1d)$$

It follows that

$$(z + w)^* = z^* + w^* \quad (4.2a)$$

$$(zw)^* = z^*w^* \quad (4.2b)$$

$$(z/w)^* = z^*/w^* \quad (4.2c)$$

$$(z^*)^* = z \quad (4.2d)$$

$$|z^*| = |z| \quad (4.2e)$$

$$1/z = z^*/(zz^*) = z^*/|z|^2 \quad (4.2f)$$

$$\frac{a + ib}{c + id} = \frac{ac + bd}{c^2 + d^2} + i \frac{bc - ad}{c^2 + d^2} \quad (4.3)$$

Euler and DeMoivre:

$$e^{i\theta} = \cos(\theta) + i \sin(\theta) \quad (4.4a)$$

$$(\cos(\theta) + i \sin(\theta))^n = \cos(n\theta) + i \sin(n\theta) \quad (4.4b)$$

$$(e^{iz})^* = e^{-iz^*}, \quad (4.5a)$$

$$(e^{i\theta})^* = e^{-i\theta}, \quad \theta \text{ real} \quad (4.5b)$$

$$z = a + ib = re^{i\theta} \text{ where } r = \sqrt{a^2 + b^2} \text{ and } \theta = \tan^{-1}(b/a).$$

Logarithm:

$$\log(z) = \log(re^{i\theta}) = \ln r + i(\theta + 2\pi k) \quad (4.6)$$

with k an integer.

Chapter 5

Useful integrals

$$\int_a^b dx f(x)g'(x) = fg|_a^b - \int_a^b dx f'g \quad (5.1)$$

Indefinite integrals:

$$\int dx \frac{1}{1+x^2} = \tan^{-1}(x) \quad (5.2a)$$

$$\int dx \frac{1}{(1+x^2)^2} = \frac{1}{2} \left[\tan^{-1}(x) + \frac{x}{1+x^2} \right] \quad (5.2b)$$

$$\int dx \frac{x}{1+x^2} = \frac{1}{2} \ln(1+x^2) \quad (5.2c)$$

$$\int dx \frac{x}{(1+x^2)^2} = -\frac{1}{2(1+x^2)} \quad (5.2d)$$

$$\int dx \frac{x^2}{1+x^2} = x - \tan^{-1}(x) \quad (5.2e)$$

$$\int dx \frac{x^2}{(1+x^2)^2} = \frac{1}{2} \tan^{-1}(x) - \frac{x}{2(1+x^2)} \quad (5.2f)$$

$$\int dx e^{-ax} = -\frac{1}{a} e^{-ax} \quad (5.3a)$$

$$\int dx x e^{-ax} = -\frac{ax+1}{a^2} e^{-ax} \quad (5.3b)$$

$$\int dx x^2 e^{-ax} = -\frac{a^2 x^2 + 2ax + 2}{a^3} e^{-ax} \quad (5.3c)$$

Some trigonometric integrals:

$$\int dx \sin^2 x = \frac{x}{2} - \frac{\sin x \cos x}{2} \quad (5.4)$$

$$\int dx \cos^2 x = \frac{x}{2} + \frac{\sin x \cos x}{2} \quad (5.5)$$

$$(5.6)$$

Assorted integrals:

$$\int dx \frac{1}{a + be^{px}} = \frac{x}{a} - \frac{1}{ap} \ln |a + be^{px}| \quad (5.7)$$

$$\int_0^\infty dx e^{-x} \ln x = -C, \quad (5.8)$$

where

$$C = \lim_{p \rightarrow \infty} \left[-\ln p + 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{p} \right] = 0.5772157 \quad (5.9)$$

is Euler's constant.

Definite integrals:

$$\int_{-\infty}^\infty dx e^{-x^2} = \sqrt{\pi} \quad (5.10)$$

$$\int_{-\infty}^\infty dx e^{-ax^2} = \frac{\sqrt{\pi}}{a^{1/2}}, \quad \operatorname{Re}[a] > 0 \quad (5.11a)$$

$$\int_{-\infty}^\infty dx x^2 e^{-ax^2} = \frac{\sqrt{\pi}}{2a^{3/2}}, \quad \operatorname{Re}[a] > 0 \quad (5.11b)$$

$$\int_{-\infty}^\infty dx x^4 e^{-ax^2} = \frac{3\sqrt{\pi}}{4a^{5/2}}, \quad \operatorname{Re}[a] > 0 \quad (5.11c)$$

$$\int_{-\infty}^\infty dx x^6 e^{-ax^2} = \frac{15\sqrt{\pi}}{8a^{7/2}}, \quad \operatorname{Re}[a] > 0 \quad (5.11d)$$

$$\int_{-\infty}^\infty dx x^p e^{-ax^2} = \frac{1 + (-1)^p}{2a^{\frac{p+1}{2}}} \Gamma\left(\frac{p+1}{2}\right), \quad \operatorname{Re}[a] > 0, \operatorname{Re}[p] > -1, \quad (5.11e)$$

see Sec. 6.1 for the Γ function.

$$\int_{-\infty}^\infty dx e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} e^{b^2/(4a)}, \quad \operatorname{Re}[a] > 0 \quad (5.12a)$$

$$\int_{-\infty}^\infty dx x e^{-ax^2+bx} = \frac{\sqrt{\pi}b}{2a^{3/2}} e^{b^2/(4a)}, \quad \operatorname{Re}[a] > 0 \quad (5.12b)$$

$$\int_{-\infty}^\infty dx x^2 e^{-ax^2+bx} = \frac{\sqrt{\pi}(2a+b^2)}{4a^{5/2}} e^{b^2/(4a)}, \quad \operatorname{Re}[a] > 0 \quad (5.12c)$$

$$\int_{-\infty}^\infty dx x^3 e^{-ax^2+bx} = \frac{\sqrt{\pi}(6ab+b^3)}{8a^{7/2}} e^{b^2/(4a)}, \quad \operatorname{Re}[a] > 0 \quad (5.12d)$$

$$\int_{-\infty}^\infty dx x^4 e^{-ax^2+bx} = \frac{\sqrt{\pi}(12a^2+12ab^2+b^4)}{16a^{9/2}} e^{b^2/(4a)}, \quad \operatorname{Re}[a] > 0 \quad (5.12e)$$

$$\int_{-\infty}^\infty dx x^6 e^{-ax^2+bx} = \frac{\sqrt{\pi}(120a^3+180a^2b^2+30ab^4+b^6)}{64a^{13/2}} e^{b^2/(4a)}, \quad \operatorname{Re}[a] > 0 \quad (5.12f)$$

The following are formally divergent but useful

$$\int_{-\infty}^{\infty} dx e^{iax^2} = \sqrt{\frac{i\pi}{a}}, \quad a \text{ real} \quad (5.13a)$$

$$\int_{-\infty}^{\infty} dx e^{i(ax^2+bx)} = \sqrt{\frac{i\pi}{a}} e^{-ib^2/(4a)}, \quad a \text{ real} \quad (5.13b)$$

$$\int_{-\infty}^{\infty} dx \frac{1}{1+bx^2} = \frac{\pi}{\sqrt{b}} \quad (5.14a)$$

$$\int_{-\infty}^{\infty} dx \frac{e^{-ax^2}}{1+bx^2} = \frac{\pi}{\sqrt{b}} e^{a/b} \text{Erfc}(\sqrt{a/b}), \quad \text{where } \text{Erfc}(x) = 1 - \text{Erf}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^x dt e^{-t^2} \quad (5.14b)$$

$$\int_{-\infty}^{\infty} dx \frac{\sin(ax)}{ax} = \frac{\pi}{a} \quad (5.15a)$$

$$\int_{-\infty}^{\infty} dx \left[\frac{\sin(ax)}{ax} \right]^2 = \frac{\pi}{a} \quad (5.15b)$$

$$\int_{-\infty}^{\infty} dx \left[\frac{\sin(ax)}{ax} \right]^3 = \frac{3\pi}{4a} \quad (5.15c)$$

$$\int_0^{\infty} dx e^{-ax} = \frac{1}{a}, \quad \text{Re}[a] > 0 \quad (5.16a)$$

$$\int_0^{\infty} dx x e^{-ax} = \frac{1}{a^2}, \quad \text{Re}[a] > 0 \quad (5.16b)$$

$$\int_0^{\infty} dx x^2 e^{-ax} = \frac{2}{a^3}, \quad \text{Re}[a] > 0 \quad (5.16c)$$

$$\int_0^{\infty} dx x^3 e^{-ax} = \frac{6}{a^4}, \quad \text{Re}[a] > 0 \quad (5.16d)$$

$$\int_0^{\infty} dx x^4 e^{-ax} = \frac{24}{a^5}, \quad \text{Re}[a] > 0 \quad (5.16e)$$

$$\int_0^{\infty} dx x^5 e^{-ax} = \frac{120}{a^6}, \quad \text{Re}[a] > 0 \quad (5.16f)$$

$$\int_0^{\infty} dx x^6 e^{-ax} = \frac{720}{a^7}, \quad \text{Re}[a] > 0 \quad (5.16g)$$

$$\int_0^{\infty} dx x^p e^{-ax} = \frac{\Gamma(p+1)}{a^{p+1}}, \quad \text{Re}[a] > 0, \text{Re}[p] > -1, \quad (5.16h)$$

see Sec. 6.1 for the Γ function.

Chapter 6

Special functions

6.1 Gamma function

Definition:

$$\Gamma(z) = \int_0^{\infty} dt t^{z-1} e^{-t} \quad (6.1)$$

$$\Gamma(n) = \int_0^{\infty} dx x^{n-1} e^{-x} = \int_0^1 dx \left(\ln \frac{1}{x} \right)^{n-1} \quad (6.2)$$

$$(6.3)$$

Properties:

$$\Gamma(z) = (z-1)\Gamma(z-1) \quad (6.4a)$$

$$\Gamma(z)\Gamma(-z) = -\frac{\pi}{z \sin(\pi z)} \quad (6.4b)$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \quad (6.4c)$$

$$(6.4d)$$

Values:

$$\Gamma(1/2) = \sqrt{\pi} \quad (6.5a)$$

If z is a positive integer $z = n = 1, 2, 3, \dots$ then

$$\Gamma(n) = (n-1)! \quad (6.6a)$$

$$\Gamma\left(n + \frac{1}{2}\right) = \sqrt{\pi} \frac{(2n-1)!!}{2^n} \quad (6.6b)$$

$$\Gamma\left(\frac{1}{2} - n\right) = \sqrt{\pi} \frac{(-1)^n 2^n}{(2n-1)!!} \quad (6.6c)$$

where $n!! = n(n-2)\dots 3 \cdot 1$, $n > 0$ odd, $n(n-2)\dots 4 \cdot 2$, $n > 0$ even, and $-1!! \equiv 1$, $0!! \equiv 1$.

6.2 Riemann zeta function

Definition:

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \quad (6.7)$$

Properties:

$$\zeta(1-z) = \frac{2}{(2\pi)^z} \cos(\pi z/2) \Gamma(z) \zeta(z) \quad (6.8a)$$

Values:

$$\zeta(2) = \frac{\pi^2}{6} \quad (6.9a)$$

$$\zeta(3) = 1.2020569032\dots \quad (6.9b)$$

6.3 Bessel and Hankel functions

Differential equation:

$$x^2 \frac{d^2 f}{dx^2} + x \frac{df}{dx} + (x^2 - \nu^2) f = 0. \quad (6.10)$$

Solutions are the Bessel functions of the first kind $J_\nu(x)$, Bessel functions of the second kind $Y_\nu(x)$, and the Hankel functions $H_\nu(x)$.

Power Series

$$J_\nu(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \Gamma(j + \nu + 1)} \left(\frac{x}{2}\right)^{\nu+2j} \quad (6.11a)$$

$$J_{-\nu}(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \Gamma(j - \nu + 1)} \left(\frac{x}{2}\right)^{-\nu+2j} \quad (6.11b)$$

Bessel functions of the second kind (Neumann functions):

$$Y_\nu(x) = \frac{J_\nu(x) \cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)} \quad (6.12)$$

Bessel functions of the third kind (Hankel functions):

$$\begin{aligned} H_\nu^{(1)}(x) &= J_\nu(x) + iY_\nu(x) \\ &= \frac{i}{\sin \nu\pi} [e^{-\nu\pi} J_\nu(x) - J_{-\nu}(x)] \end{aligned} \quad (6.13a)$$

$$\begin{aligned} H_\nu^{(2)}(x) &= J_\nu(x) - iY_\nu(x) \\ &= \frac{-i}{\sin \nu\pi} [e^{\nu\pi} J_\nu(x) - J_{-\nu}(x)] \end{aligned} \quad (6.13b)$$

$$J_\nu(x) = \frac{1}{2} [H_\nu^{(1)}(x) + H_\nu^{(2)}(x)] \quad (6.14a)$$

$$J_{-\nu}(x) = \frac{1}{2} [e^{\nu\pi} H_\nu^{(1)}(x) + e^{-\nu\pi} H_\nu^{(2)}(x)] \quad (6.14b)$$

Generating function:

$$e^{z(t-1/t)/2} = \sum_{n=-\infty}^{\infty} t^n J_n(z) \quad n \text{ takes on integer values} \quad (6.15a)$$

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \left(\frac{x}{2}\right)^{n+2k} \quad \text{for } n \text{ integer} \quad (6.15b)$$

$$J_n(x) = \frac{1}{\pi} \int_0^\pi d\theta \cos(n\theta - x \sin \theta) \quad \text{for } n \text{ integer} \quad (6.15c)$$

Orthogonality:

$$\int_0^\infty dx x J_\nu(ax) J_\nu(bx) = \frac{1}{a} \delta(|a| - |b|) \quad (6.16)$$

Recurrence relations (valid for both $J_\nu(x)$ and $H_\nu(x)$):

$$\frac{d}{dx}(x^\nu J_\nu(x)) = x^\nu J_{\nu-1}(x) \quad (6.17a)$$

$$\frac{2\nu}{x} J_\nu(x) = J_{\nu+1}(x) + J_{\nu-1}(x) \quad (6.17b)$$

$$\frac{dJ_\nu(x)}{dx} = J_{\nu-1}(x) - \frac{\nu}{x} J_\nu(x) \quad (6.17c)$$

$$\frac{dJ_\nu(x)}{dx} = \frac{\nu}{x} J_\nu(x) - J_{\nu+1}(x) \quad (6.17d)$$

$$\frac{dJ_\nu(x)}{dx} = \frac{1}{2} [J_{\nu-1}(x) - J_{\nu+1}(x)] \quad (6.17e)$$

Symmetries:

$$J_{-n}(x) = (-1)^n J_n(x) \quad \text{for } n \text{ integer.} \quad (6.18a)$$

Particular values:

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x \quad (6.19a)$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x \quad (6.19b)$$

Asymptotic values:

For $x \ll 1$

$$J_\nu(x) \rightarrow \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu \quad (6.20a)$$

$$Y_0(x) \rightarrow \frac{2}{\pi} [\ln(x/2) + 0.5772\dots] \quad (6.20b)$$

$$Y_\nu(x) \rightarrow -\frac{\Gamma(\nu)}{\pi} \left(\frac{2}{x}\right)^\nu \quad (6.20c)$$

For $x \gg 1$

$$J_\nu(x) \rightarrow \sqrt{\frac{2}{\pi x}} \cos(x - \nu\pi/2 - \pi/4) \quad (6.21a)$$

$$Y_\nu(x) \rightarrow \sqrt{\frac{2}{\pi x}} \sin(x - \nu\pi/2 - \pi/4) \quad (6.21b)$$

$$H_\nu^{(1)}(x) \rightarrow \sqrt{\frac{2}{\pi x}} e^{i(x - \nu\pi/2 - \pi/4)} \quad (6.21c)$$

$$H_\nu^{(2)}(x) \rightarrow \sqrt{\frac{2}{\pi x}} e^{-i(x - \nu\pi/2 - \pi/4)} \quad (6.21d)$$

Sums:

$$e^{iz \cos \theta} = \sum_{n=-\infty}^{\infty} i^n J_n(z) e^{m\theta} \quad n \text{ takes on integer values} \quad (6.22a)$$

$$1 = \sum_{n=-\infty}^{\infty} J_n(x) \quad n \text{ takes on integer values} \quad (6.22b)$$

$$J_n(x+y) = \sum_{m=-\infty}^{\infty} J_m(x) J_{n-m}(y) \quad (6.22c)$$

6.4 Delta function

Definition:

$$\delta(x) = 0 \quad x \neq 0, \quad (6.23a)$$

$$= \infty \quad x = 0, \quad (6.23b)$$

$$\int_{-\infty}^{\infty} dx f(x) \delta(x) = f(0) \quad (6.23c)$$

Properties:

$$\int_{-\infty}^{\infty} dx \delta(x) = 1 \quad (6.24a)$$

$$\int_{-\infty}^{\infty} dx f(x) \delta(x - x_0) = f(x_0) \quad (6.24b)$$

$$\delta(x - x_0) = \delta(x_0 - x) \quad (6.25a)$$

$$\delta(ax) = \frac{1}{|a|} \delta(x) \quad (6.25b)$$

$$\delta(g(x)) = \sum_j \frac{1}{|g'(x_j)|} \delta(x - x_j), \quad \text{where } x_j \text{ are the roots of } g \quad (6.25c)$$

$$\frac{d^n}{dx^n} \delta(x) = \frac{(-1)^n n!}{x^n} \delta(x) \quad (6.25d)$$

Representations:

$$\delta(x) = \lim_{\sigma \rightarrow 0} \frac{1}{\sqrt{2\pi} \sigma} e^{-x^2/2\sigma^2} \quad (6.26a)$$

$$= \lim_{\sigma \rightarrow 0} \frac{1}{\pi} \frac{\sigma}{x^2 + \sigma^2} \quad (6.26b)$$

$$= \lim_{\sigma \rightarrow 0} \frac{1}{\pi x} \sin(x/\sigma) \quad (6.26c)$$

$$= \lim_{\sigma \rightarrow 0} \frac{\sigma}{2\pi} \frac{\sin^2 \frac{x}{2\sigma}}{(x/2)^2} \quad (6.26d)$$

$$\delta(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\sqrt{i\pi\varepsilon}} e^{i\frac{x^2}{\varepsilon}} \quad (6.27)$$

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx} \quad (6.28)$$

If $u_n(x)$ form a complete set then

$$\delta(x - x') = \sum_n u_n(x) u_n^*(x') \quad (6.29)$$

Multiple dimensions:

Cartesian coordinates:

$$\delta(\mathbf{r}) = \delta(x)\delta(y)\delta(z) \quad (6.30a)$$

Spherical coordinates:

$$\delta(\mathbf{r} - \mathbf{r}') = \frac{1}{r^2} \delta(r - r') \delta(\cos \theta - \cos \theta') \delta(\phi - \phi') \quad (6.31a)$$

$$= \frac{1}{r^2} \delta(r - r') \frac{\delta(\theta - \theta')}{\sin \theta} \delta(\phi - \phi') \quad (6.31b)$$

$$\nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -4\pi \delta(\mathbf{r} - \mathbf{r}') \quad (6.31c)$$

Chapter 7

Fourier transforms

7.1 Transforms on the full line

Transform pair:

$$f(x) = \mathcal{F}^{-1}[g(k)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk g(k) e^{ikx} \quad (7.1a)$$

$$g(k) = \mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx f(x) e^{-ikx} \quad (7.1b)$$

Plancherel:

$$\int_{-\infty}^{\infty} dx f_1^*(x) f_2(x) = \int_{-\infty}^{\infty} dk g_1^*(k) g_2(k) \quad (7.2a)$$

$$\int_{-\infty}^{\infty} dx |f(x)|^2 = \int_{-\infty}^{\infty} dk |g(k)|^2 \quad (7.2b)$$

$$(7.2c)$$

Convolution:

$$f_1(x) * f_2(x) \equiv \int_{-\infty}^{\infty} dx' f_1(x') f_2(x - x') \quad (7.3a)$$

$$\mathcal{F}[f_1(x) * f_2(x)] = \sqrt{2\pi} g_1(k) g_2(k) \quad (7.3b)$$

Correlation:

$$f_1(x) \star f_2(x) \equiv [f_1(-x)]^* * f_2(x) = \int_{-\infty}^{\infty} dx' [f_1(-x')]^* f_2(x - x') \quad (7.4a)$$

$$\mathcal{F}[f_1(x) \star f_2(x)] = \sqrt{2\pi} [g_1(k)]^* g_2(k) \quad (7.4b)$$

Multiple dimensions:

$$f(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d\mathbf{k} g(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} \quad (7.5a)$$

$$g(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d\mathbf{r} f(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} \quad (7.5b)$$

Quantum mechanics version:

$$f(\mathbf{r}) = \frac{1}{(2\pi\hbar)^{3/2}} \int_{-\infty}^{\infty} d\mathbf{p} g(\mathbf{p}) e^{(i/\hbar)\mathbf{p}\cdot\mathbf{r}} \quad (7.6a)$$

$$g(\mathbf{p}) = \frac{1}{(2\pi\hbar)^{3/2}} \int_{-\infty}^{\infty} d\mathbf{r} f(\mathbf{r}) e^{-(i/\hbar)\mathbf{p}\cdot\mathbf{r}} \quad (7.6b)$$

Transform pairs:

When $f(x) \leftrightarrow g(k)$ then:

$$f(x - x_0) \leftrightarrow e^{-ikx_0} g(k) \quad (7.7a)$$

$$e^{ik_0x} f(x) \leftrightarrow g(k - k_0) \quad (7.7b)$$

$$\frac{df(x)}{dx} \leftrightarrow ikg(k) \quad (7.7c)$$

$$xf(x) \leftrightarrow i \frac{dg(k)}{dk} \quad (7.7d)$$

$$x \frac{df(x)}{dx} \leftrightarrow -g(k) - k \frac{dg(k)}{dk} \quad (7.7e)$$

$$\delta(x - x_0) \leftrightarrow \frac{1}{\sqrt{2\pi}} e^{-ikx_0} \quad (7.8a)$$

$$e^{ik_0x} \leftrightarrow \sqrt{2\pi} \delta(k - k_0) \quad (7.8b)$$

$$e^{-ax^2+bx} \leftrightarrow \frac{1}{\sqrt{2a}} e^{(b-ik)^2/(4a)} \quad (7.8c)$$

$$xe^{-ax^2+bx} \leftrightarrow \frac{b-ik}{(2a)^{3/2}} e^{(b-ik)^2/(4a)} \quad (7.8d)$$

$$x^2 e^{-ax^2+bx} \leftrightarrow \frac{2a + (b-ik)^2}{(2a)^{5/2}} e^{(b-ik)^2/(4a)} \quad (7.8e)$$

$$\frac{1}{1+x^2} \leftrightarrow \sqrt{\frac{\pi}{2}} e^{-|k|} \quad (7.9a)$$

$$\frac{1}{(1+x^2)^2} \leftrightarrow \sqrt{\frac{\pi}{8}} (1+|k|) e^{-|k|} \quad (7.9b)$$

7.2 Transforms on the half line

For functions defined on the half line there are sin and cos transform pairs:

$$f(x) = \mathcal{F}_c^{-1}[g(k)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} dk g(k) \cos(kx) \quad (7.10a)$$

$$g(k) = \mathcal{F}_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} dx f(x) \cos(kx) \quad (7.10b)$$

$$f(x) = \mathcal{F}_s^{-1}[g(k)] = \sqrt{\frac{2}{\pi}} \int_0^\infty dk g(k) \sin(kx) \quad (7.11a)$$

$$g(k) = \mathcal{F}_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty dx f(x) \sin(kx) \quad (7.11b)$$

Chapter 8

Orthogonal polynomials

8.1 Hermite polynomials

Differential equation:

$$\frac{d^2 f}{dx^2} - 2x \frac{df}{dx} + 2nf = 0. \quad (8.1)$$

Particular solutions for n integer are the Hermite polynomials $H_n(x)$.

Generating function:

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) \quad (8.2a)$$

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \quad (8.2b)$$

Recurrence relations:

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \quad (8.3a)$$

$$H'_n(x) = 2nH_{n-1}(x) \quad (8.3b)$$

Particular values:

$$H_n(-x) = (-1)^n H_n(x) \quad (8.4)$$

$$H_0(x) = 1 \quad (8.5a)$$

$$H_1(x) = 2x \quad (8.5b)$$

$$H_2(x) = 4x^2 - 2 \quad (8.5c)$$

$$H_3(x) = 8x^3 - 12x \quad (8.5d)$$

$$H_4(x) = 16x^4 - 48x^2 + 12 \quad (8.5e)$$

$$H_5(x) = 32x^5 - 160x^3 + 120x \quad (8.5f)$$

Orthogonality relations:

$$u_n(x) = \sqrt{\frac{1}{\pi^{1/2} n! 2^n a}} H_n(x/a) e^{-x^2/(2a^2)} \quad a \text{ is a scaling parameter which can be 1} \quad (8.6)$$

$$\int_{-\infty}^{\infty} dx u_m(x) u_n(x) = \delta_{mn} \quad (8.7a)$$

$$\int_{-\infty}^{\infty} dx u_n(x) u'_m(x) = \frac{1}{a} \sqrt{\frac{n+1}{2}} \delta_{m,n+1} - \frac{1}{a} \sqrt{\frac{n}{2}} \delta_{m,n-1} \quad (8.7b)$$

$$\int_{-\infty}^{\infty} dx u_m(x) x u_n(x) = a \sqrt{\frac{n+1}{2}} \delta_{m,n+1} + a \sqrt{\frac{n}{2}} \delta_{m,n-1} \quad (8.7c)$$

$$\int_{-\infty}^{\infty} dx u_m(x) x^2 u_n(x) = a^2 \frac{2n+1}{2} \delta_{mn} + a^2 \frac{\sqrt{(n+2)(n+1)}}{2} \delta_{m,n+2} + a^2 \frac{\sqrt{n(n-1)}}{2} \delta_{m,n-2} \quad (8.7d)$$

Delta function expansion:

$$\delta(x-y) = \frac{e^{-x^2}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{H_n(x) H_n(y)}{2^n n!} \quad (8.8)$$

Fourier transform:

$$\mathcal{F}[H_n(x/a) e^{-x^2/(2a^2)}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx H_n(x/a) e^{-x^2/(2a^2)} e^{-ikx} = (-i)^n a H_n(ak) e^{-a^2 k^2/2} \quad (8.9a)$$

$$\mathcal{F}[u_n(x/a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx u_n(x/a) e^{-ikx} = (-i)^n u_n(ak) \quad (8.9b)$$

8.2 Laguerre polynomials

Differential equation:

$$x \frac{d^2 f}{dx^2} + (1-x) \frac{df}{dx} + n f = 0. \quad (8.10)$$

Particular solutions for n integer are the Laguerre polynomials $L_n(x)$.

Generating function:

$$\frac{1}{1-z} e^{-xz/(1-z)} = \sum_{n=0}^{\infty} L_n(x) z^n \quad (8.11a)$$

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} (x^n e^{-x}) \quad (8.11b)$$

Recurrence relations:

$$(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x) \quad (8.12a)$$

$$xL'_n(x) = nL_n(x) - nL_{n-1}(x) \quad (8.12b)$$

Particular values:

$$L_0(x) = 1 \quad (8.13a)$$

$$L_1(x) = 1 - x \quad (8.13b)$$

$$L_2(x) = \frac{1}{2}(x^2 - 4x + 2) \quad (8.13c)$$

$$L_3(x) = \frac{1}{6}(-x^3 + 9x^2 - 18x + 6) \quad (8.13d)$$

$$L_4(x) = \frac{1}{24}(x^4 - 16x^3 + 72x^2 - 96x + 24) \quad (8.13e)$$

Orthogonality relations:

$$\int_0^\infty dx L_m(x)L_n(x)e^{-x} = \delta_{mn} \quad (8.14a)$$

$$\int_0^\infty dx x [L_n(x)]^2 e^{-x} = (2n + 1) \quad (8.14b)$$

8.3 Associated Laguerre polynomials

Differential equation:

$$x \frac{d^2 f}{dx^2} + (k + 1 - x) \frac{df}{dx} + n f = 0. \quad (8.15)$$

Particular solutions for n, k integer are the associated Laguerre polynomials $L_n^k(x)$.

Generating function:

$$\frac{1}{(1-z)^{k+1}} e^{-xz/(1-z)} = \sum_{n=0}^{\infty} L_n^k(x) z^n \quad (8.16a)$$

$$\frac{1}{1-z} e^{\frac{-xz+u}{1-z}} = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{L_n^k(x) z^n u^k}{k!} \quad (8.16b)$$

$$L_n^k(x) = \frac{x^{-k} e^x}{n!} \frac{d^n}{dx^n} (x^{n+k} e^{-x}) \quad (8.16c)$$

$$L_n^k(x) = (-1)^k \frac{d^k}{dx^k} L_{n+k}(x) \quad (8.16d)$$

Recurrence relations:

$$L_n^k(x) = L_{n-1}^k(x) + L_n^{k-1}(x) \quad (8.17a)$$

$$(n+1)L_{n+1}^k(x) = (2n+k+1-x)L_n^k(x) - (n+k)L_{n-1}^k(x) \quad (8.17b)$$

$$x \frac{dL_n^k(x)}{dx} = nL_n^k(x) - (n+k)L_{n-1}^k(x) \quad (8.17c)$$

Particular values:

$$L_0^k(x) = 1 \quad (8.18a)$$

$$L_1^k(x) = 1 + k - x \quad (8.18b)$$

$$L_2^k(x) = \frac{1}{2}(x^2 - 2(k+2)x + (k+1)(k+2)) \quad (8.18c)$$

$$L_3^k(x) = \frac{1}{6}(-x^3 + 3(k+3)x^2 - 3(k+3)(k+2)x + (k+3)(k+2)(k+1)) \quad (8.18d)$$

$$L_n^k(0) = L_n^k(0) = \binom{n+k}{n} = \frac{(n+k)!}{n!k!} \quad (8.18e)$$

Orthogonality relations:

$$\int_0^\infty dx L_m^k(x) L_n^k(x) e^{-x} x^k = \frac{(n+k)!}{n!} \delta_{mn} \quad (8.19a)$$

$$\int_0^\infty dx [L_n^k(x)]^2 e^{-x} x^{k+1} = \frac{(n+k)!}{n!} (2n+k+1) \quad (8.19b)$$

8.4 Legendre polynomials

Differential equation:

$$(1-x^2) \frac{d^2 f}{dx^2} - 2x \frac{df}{dx} + l(l+1)f = 0. \quad (8.20)$$

Particular solutions for l integer are the Legendre polynomials $P_l(x)$.

Generating function:

$$\frac{1}{\sqrt{t^2 - 2xt + 1}} = \sum_{l=0}^{\infty} P_l(x) t^l \quad (8.21a)$$

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l \quad (8.21b)$$

Recurrence relations:

$$(l+1)P_{l+1}(x) = (2l+1)xP_l(x) - lP_{l-1}(x) \quad (8.22a)$$

$$lP_l(x) = xP_l'(x) - P_{l-1}'(x) \quad (8.22b)$$

$$lP_l(x) = lxP_{l-1}(x) + (x^2 - 1)P_{l-1}'(x) \quad (8.22c)$$

$$(l+1)P_l(x) = P_{l+1}'(x) - xP_l'(x) \quad (8.22d)$$

Particular values:

$$P_l(-x) = (-1)^l P_l(x) \quad (8.23)$$

$$P_l(1) = 1 \quad (8.24a)$$

$$P_l(-1) = (-1)^l \quad (8.24b)$$

$$P_l(0) = \frac{\sqrt{\pi}}{\Gamma(\frac{1-l}{2})\Gamma(1+\frac{l}{2})} \quad (8.24c)$$

$$P_0(x) = 1 \quad (8.25a)$$

$$P_1(x) = x \quad (8.25b)$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \quad (8.25c)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x) \quad (8.25d)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \quad (8.25e)$$

Orthogonality relations:

$$u_l(x) = \sqrt{\frac{2l+1}{2}} P_l(x) \quad (8.26)$$

$$\int_{-1}^1 dx u_l(x) u_{l'}(x) = \delta_{ll'} \quad (8.27a)$$

$$\sum_{l=0}^{\infty} u_l(x) u_l(x') = \delta(x - x') \quad (8.27b)$$

Sums:

$$\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = \sum_{l=0}^{\infty} \frac{r_{<}^l}{r_{>}^{l+1}} P_l(\cos \gamma) \quad (8.28)$$

where $\cos \gamma = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos(\phi_1 - \phi_2)$.

8.5 Associated Legendre polynomials

Differential equation:

$$(1-x^2) \frac{d^2 f}{dx^2} - 2x \frac{df}{dx} + \left[l(l+1) - \frac{m^2}{1-x^2} \right] f = 0. \quad (8.29)$$

Particular solutions for l, m integer with $-l \leq m \leq l$ are the associated Legendre polynomials $P_l^m(x)$.

Generating function:

$$\frac{1}{\sqrt{t^2 - 2t(x + z\sqrt{1-x^2}) + 1}} = \sum_{l=0}^{\infty} \sum_{m=0}^l P_l^m(x) \frac{(-1)^m t^l z^m}{m!} \quad (8.30a)$$

$$P_l^m(x) = (-1)^m \frac{(1-x^2)^{m/2}}{2^l l!} \frac{d^{l+m}}{dx^{l+m}} (x^2 - 1)^l, \quad \text{all } m \quad (8.30b)$$

$$P_l^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x), \quad m > 0 \quad (8.30c)$$

$$(8.30d)$$

Recurrence relation (there are many more):

$$(2l + 1)xP_l^m(x) = (l - m + 1)P_{l+1}^m(x) + (l + m)P_{l-1}^m(x) \quad (8.31a)$$

Particular values:

$$P_l^0(x) = P_l(x) \quad (8.32a)$$

$$P_l^{-m}(x) = (-1)^m \frac{(l - m)!}{(l + m)!} P_l^m(x) \quad (8.32b)$$

$$P_l^l(x) = (-1)^l (2l - 1)!! (1 - x^2)^{l/2} \quad (8.32c)$$

$$P_{l+1}^l(x) = (2l + 1)xP_l^l(x) \quad (8.32d)$$

$$P_0^0(x) = 1 \quad (8.33a)$$

$$P_1^{-1}(x) = \frac{\sqrt{1 - x^2}}{2}, \quad P_1^0(x) = x, \quad P_1^1(x) = -\sqrt{1 - x^2} \quad (8.33b)$$

$$P_2^{-2}(x) = \frac{1 - x^2}{8}, \quad P_2^{-1}(x) = \frac{x\sqrt{1 - x^2}}{2}, \quad P_2^0(x) = \frac{3x^2 - 1}{2}, \quad (8.33c)$$

$$P_2^1(x) = -3x\sqrt{1 - x^2}, \quad P_2^2(x) = -3(x^2 - 1) \quad (8.33d)$$

Orthogonality relations:

$$u_l^m(x) = \sqrt{\frac{2l + 1}{2}} P_l^m(x) \quad (8.34)$$

$$\int_{-1}^1 dx u_l^m(x) u_{l'}^{m'}(x) = \frac{(l + m)!}{(l - m)!} \delta_{l,l'} \quad (8.35a)$$

$$\int_{-1}^1 dx u_l^m(x) u_l^{m'}(x) \frac{1}{1 - x^2} = \frac{2l + 1}{2} \frac{(l + m)!}{m(l - m)!} \delta_{m,m'} \quad (8.35b)$$

8.6 Spherical Harmonics

Differential equation:

$$\left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + l(l + 1) \right] f = 0 \quad (8.36)$$

Particular solutions are $Y_{lm}(\theta, \phi)$ with integer $l \geq 0$, $-l \leq m \leq l$. The notation $Y_l^m(\theta, \phi) = Y_{lm}(\theta, \phi)$ is also used.

Definition:

$$Y_{lm}(\theta, \phi) = \left[\frac{2l + 1}{4\pi} \frac{(l - m)!}{(l + m)!} \right]^{1/2} P_l^m(\cos \theta) e^{im\phi} \quad (8.37)$$

Symmetries:

$$Y_{l,-m}(\theta, \phi) = (-1)^m [Y_{lm}(\theta, \phi)]^* \quad (8.38a)$$

$$Y_{lm}(-\theta, \phi) = (-1)^m Y_{lm}(\theta, \phi) \quad (8.38b)$$

$$Y_{lm}(\theta, -\phi) = (-1)^m Y_{l,-m}(\theta, \phi) \quad (8.38c)$$

$$Y_{lm}(-\theta, -\phi) = Y_{l,-m}(\theta, \phi) \quad (8.38d)$$

$$Y_{lm}(\pi - \theta, \pi + \phi) = (-1)^l Y_{lm}(\theta, \phi) \quad (8.39a)$$

$$Y_{lm}(\pi - \theta, \phi) = (-1)^{l+m} Y_{lm}(\theta, \phi) \quad (8.39b)$$

$$Y_{lm}(\theta, \pi + \phi) = (-1)^m Y_{lm}(\theta, \phi) \quad (8.39c)$$

Integrals over the entire solid angle with $d\Omega = \sin\theta d\theta d\phi$

$$\int d\Omega [Y_{lm}(\theta, \phi)]^* Y_{l'm'}(\theta, \phi) = \delta_{ll'} \delta_{mm'} \quad (8.40a)$$

$$\int d\Omega Y_{lm}(\theta, \phi) Y_{l'm'}(\theta, \phi) = (-1)^{m'} \delta_{ll'} \delta_{-m,m'} \quad (8.40b)$$

$$\int d\Omega Y_{l_1 m_1} Y_{l_2 m_2} Y_{l_3 m_3} = \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \quad (8.40c)$$

$$= (-1)^{m_3} \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l_3+1)}} C_{l_1 0 l_2 0}^{l_3 0} C_{l_1 - m_1 l_2 - m_2}^{l_3 m_3} \quad (8.40d)$$

$$\langle l_3 m_3 | Y_{l_2 m_2} | l_1 m_1 \rangle = (-1)^{l_1 + l_2 - l_3} \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l_3+1)}} C_{l_1 0 l_2 0}^{l_3 0} C_{l_1 m_1 l_2 m_2}^{l_3 m_3} \quad (8.40e)$$

Sums:

$$\sum_{m=-l}^l |Y_{lm}(\theta, \phi)|^2 = \frac{2l+1}{4\pi} \quad (8.41a)$$

$$\sum_{m=-l}^l m |Y_{lm}(\theta, \phi)|^2 = 0 \quad (8.41b)$$

$$\sum_{m=-l}^l m^2 |Y_{lm}(\theta, \phi)|^2 = \frac{(2l+1)(l+1)l}{8\pi} \sin^2 \theta \quad (8.41c)$$

$$\sum_{l=0}^{\infty} \sum_{m=-l}^l [Y_{lm}(\theta, \phi)]^* Y_{lm}(\theta', \phi') = \delta(\cos\theta - \cos\theta') \delta(\phi - \phi') \quad (8.42)$$

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^l Y_{lm}^*(\theta_2, \phi_2) Y_{lm}(\theta_1, \phi_1) \quad (8.43)$$

where $\cos \gamma = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos(\phi_1 - \phi_2)$.

Rank l spherical tensor

$$C_{lm}(\theta, \phi) = \left(\frac{4\pi}{2l+1} \right)^{1/2} Y_{lm}(\theta, \phi) \quad (8.44)$$

and

$$\begin{aligned} P_l(\cos \gamma) &= \sum_{m=-l}^l C_{lm}^*(\theta_2, \phi_2) C_{lm}(\theta_1, \phi_1) \\ &= \sum_{m=-l}^l (-1)^m C_{l-m}(\theta_2, \phi_2) C_{lm}(\theta_1, \phi_1) \\ &= (\mathbf{C}_l(1) \cdot \mathbf{C}_l(2)) \end{aligned} \quad (8.45)$$

$$\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{1}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}^*(\theta_2, \phi_2) Y_{lm}(\theta_1, \phi_1)$$

Solutions of the equation $\nabla^2 f(r, \theta, \phi) = 0$ are of the form (solid harmonics)

$$r^l Y_{lm}(\theta, \phi) \quad (8.46a)$$

or

$$\frac{1}{r^{l+1}} Y_{lm}(\theta, \phi) \quad (8.46b)$$

Chapter 9

Angular momentum algebra

9.1 Clebsch-Gordan coefficients and symmetry properties

There exist a number of explicit expressions for the Clebsch-Gordan coefficients. An expression due to Wigner is

$$C_{a\alpha b\beta}^{c\gamma} = \delta_{\gamma, \alpha+\beta} \Delta(abc) \left[\frac{(c+\gamma)!(c-\gamma)!(2c+1)}{(a+\alpha)!(a-\alpha)!(b+\beta)!(b-\beta)!} \right]^{1/2} \\ \times \sum_z \frac{(-1)^{b+\beta+z} (c+b+\alpha-z)!(a-\alpha+z)!}{z!(c-a+b-z)!(c+\gamma-z)!(a-b-\gamma+z)!} \quad (9.1)$$

where

$$\Delta(abc) = \sqrt{\frac{(a+b-c)!(a-b+c)!(-a+b+c)!}{(a+b+c+1)!}}$$

and in the summation z assumes all integer values for which the factorial arguments are nonnegative. The Clebsch-Gordan coefficients satisfy the following unitarity relations

$$\sum_{m_1 m_2} C_{j_1 m_1 j_2 m_2}^{j m} C_{j_1 m_1 j_2 m_2}^{j' m'} = \delta_{j j'} \delta_{m m'} \quad (9.2a)$$

$$\sum_{j m} C_{j_1 m_1 j_2 m_2}^{j m} C_{j_1 m'_1 j_2 m'_2}^{j m} = \delta_{m_1 m'_1} \delta_{m_2 m'_2}. \quad (9.2b)$$

There are a large number of symmetry relations involving permutations of indices. Some of them are:

$$C_{a\alpha b\beta}^{c\gamma} = (-1)^{a+b-c} C_{b\beta a\alpha}^{c\gamma} = (-1)^{a-\alpha} \sqrt{\frac{2c+1}{2b+1}} C_{a\alpha c-\gamma}^{b-\beta} = (-1)^{a-\alpha} \sqrt{\frac{2c+1}{2b+1}} C_{c\gamma a-\alpha}^{b\beta} \\ = (-1)^{b+\beta} \sqrt{\frac{2c+1}{2a+1}} C_{c-\gamma b\beta}^{a-\alpha} = (-1)^{b+\beta} \sqrt{\frac{2c+1}{2a+1}} C_{b-\beta c\gamma}^{a\alpha} \quad (9.3)$$

$$C_{a\alpha b\beta}^{c\gamma} = (-1)^{a+b-c} C_{a-\alpha b-\beta}^{c-\gamma} \quad (9.4)$$

When one of the momenta is zero:

$$C_{a\alpha b\beta}^{00} = (-1)^{a-\alpha} \frac{\delta_{ab}\delta_{\alpha-\beta}}{\sqrt{2a+1}} \quad (9.5a)$$

$$C_{a\alpha 00}^{c\gamma} = \delta_{ac}\delta_{\alpha\gamma} \quad (9.5b)$$

When the third momentum is the maximum possible:

$$C_{a\alpha b\beta}^{a+b\alpha+\beta} = \left[\frac{(2a)!(2b)!(a+b+\alpha+\beta)!(a+b-\alpha-\beta)!}{(2a+2b)!(a+\alpha)!(a-\alpha)!(b+\beta)!(b-\beta)!} \right]^{1/2} \quad (9.6)$$

When all the m 's are zero:

$$C_{a0b0}^{a+b0} = \frac{(a+b)!}{a!b!} \left[\frac{(2a)!(2b)!}{(2a+2b)!} \right]^{1/2} \quad (9.7a)$$

$$C_{a0b0}^{a-b0} = (-1)^b \frac{a!}{b!(a-b)!} \left[\frac{(2b)!(2a-2b+1)!}{(2a+1)!} \right]^{1/2} \quad (9.7b)$$

In particular when $b = 1$

$$C_{j010}^{j\pm 10} = \pm \sqrt{\frac{j_{\max}}{2j+1}} \quad (9.8)$$

where j_{\max} is the larger of $j, j \pm 1$.

The coefficients for the cases when one of the momenta is $1/2$ or 1 are often needed in atomic calculations. The $b = 1/2$ cases are:

$$C_{a\alpha 1/2\pm 1/2}^{a+1/2\alpha\pm 1/2} = \sqrt{\frac{a \pm \alpha + 1}{2a+1}} \quad (9.9a)$$

$$C_{a\alpha 1/2\pm 1/2}^{a-1/2\alpha\pm 1/2} = \mp \sqrt{\frac{a \mp \alpha}{2a+1}} \quad (9.9b)$$

The $b = 1$ cases are:

$$C_{a\alpha 10}^{a+1\alpha} = \sqrt{\frac{(a+\alpha+1)(a-\alpha+1)}{(2a+1)(a+1)}}, \quad C_{a\alpha 1\pm 1}^{a+1\alpha\pm 1} = \sqrt{\frac{(a \pm \alpha + 1)(a \pm \alpha + 2)}{2(2a+1)(a+1)}} \quad (9.10a)$$

$$C_{a\alpha 10}^{a\alpha} = \frac{\alpha}{\sqrt{a(a+1)}}, \quad C_{a\alpha 1\pm 1}^{a\alpha\pm 1} = \mp \sqrt{\frac{(a \pm \alpha + 1)(a \mp \alpha)}{2a(a+1)}} \quad (9.10b)$$

$$C_{a\alpha 10}^{a-1\alpha} = -\sqrt{\frac{(a+\alpha)(a-\alpha)}{a(2a+1)}}, \quad C_{a\alpha 1\pm 1}^{a-1\alpha\pm 1} = \sqrt{\frac{(a \mp \alpha - 1)(a \mp \alpha)}{2a(2a+1)}} \quad (9.10c)$$

The $3j$ symbols are given by

$$\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} = (-1)^{2a+c+\gamma} \frac{1}{\sqrt{2c+1}} C_{a-\alpha b-\beta}^{c\gamma}. \quad (9.11)$$