Introduction to the Fourier Transform

I. Fourier Series

The Fourier transform is the name of a very useful method to study waves, of many different sorts, and also to solve several kinds of linear differential equations. It is an extension of the Fourier series: the Fourier series is a way to "decompose" a periodic function into a sum of parts, each of which have periods that are fractions of the full period; we refer to these parts as higher-frequency modes, or higher harmonics. (Of course the frequency is the inverse of the period.) The Fourier transform is a way to "decompose" an arbitrary function, usually not periodic, into its various periodic modes, usually requiring an infinite number of periods.

If one adds together just two cosine functions, with different frequencies, the result is a product of a cosine with the difference and the sum of the original two frequencies:

$$f(x) = A\cos(k_1 x) + A\cos(k_2 x) = 2A\cos[\tfrac{1}{2}(k_1 - k_2)x]\cos[\tfrac{1}{2}(k_1 + k_2)x]\ . \tag{1.1}$$

If the original two frequencies are near one another, then half the sum, i.e., the average is still near the original frequencies, so that it still oscillates in about the same way. However, its amplitude is now "modulated" by the other cosine function, with the difference of the two frequencies, which is therefore quite small, generating a very large period; i.e., the amplitude now oscillates slowly. This slowly-oscillating amplitude function is usually referred to as an *envelope* for the more-rapidly-oscillating part of the wave.

The use of Fourier series is a way to take infinite sums of waves with different periods, rather than just a sum of two, so that one may generate reasonably arbitrary, but periodic, functions. To do this we suppose that we have been given a function, $f = f(x)$, which is periodic, with period L:

$$\begin{gathered} f = f(x) \text{ is a periodic function, with period } L \\ \Longleftrightarrow \\ f(x) = f(x \pm L) = f(x + mL)\,, \quad \text{for any integer } m\ . \end{gathered} \tag{1.2}$$

The theory of Fourier series then tells us that we may find

sequences of numbers $\{a_n \mid n = 0, 1, 2, \ldots\}$ and $\{b_n \mid n = 0, 1, 2, \ldots\}$ such that

$$f(x) = \sum_{n=0}^{\infty} \{a_n \cos k_n x + b_n \sin k_n x\} \ , \qquad k_n \equiv 2n\pi/L \ , \tag{1.3}$$

where it is in fact true that the equality is not necessarily exactly so at the (two) ends of the period L, but only just at those two points. An important part of the underlying rationale for the existence, and use, of such an expression is that all the various wave numbers k_n are chosen so that all possible integer multiples of the lowest frequency, $2\pi/L$ are included, or, equivalently, so that all possible integer fractions of the original period, L, are included. However, it is also valuable to rewrite this expression using deMoivre's theorem about complex-valued exponentials, namely

$$e^{iz} = \cos z + i \sin z \qquad \Longleftrightarrow \qquad \begin{cases} 2\cos z = e^{+iz} + e^{-iz} \ , \\ 2i \sin z = e^{+iz} - e^{-iz} \ . \end{cases} \tag{1.4}$$

By setting $c_n = \frac{1}{2}(a_n - ib_n)$ for positive values of n and $c_n = \frac{1}{2}(a_{-n} + ib_{-n})$ for negative values of n, we may rewrite our expression using complex exponentials instead:

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n \, e^{ik_n x} \ . \tag{1.5}$$

The apparent discrepancy when $n = 0$, as to whether it is positive or negative, disappears when we note that $\sin k_0 x = \sin 0 = 0$, so that we ought to just go ahead and choose $b_0 = 0$, so that $c_0 = \frac{1}{2} a_0$ according to both the definition for positive values of n and that for negative values. The equation may be solved, to determine the various quantities c_n, which are referred to as the Fourier coefficients of the periodic function f:

$$c_n = \frac{1}{L} \int_{-\frac{1}{2}L}^{+\frac{1}{2}L} dx \ f(x) \, e^{-ik_n x} \ . \tag{1.6}$$

To verify this we should show that the two equations, (1.5) and (1.6), are consistent, **in both directions.** We begin by inserting the claimed definition for $f(x)$ into Eq. (1.6), but noting that the index n in Eq. (1.5) for $f(x)$ is just a "dummy" variable, i.e., a variable indicating a summation, so that when we insert this value for $f(x)$ into Eq. (1.6) we must change the name of that index, from n to something else, so we do not confuse it with the actual index n that is on the left-hand side of the equation; we agree to call it m and then have the following, where we assume that the functions involved are sufficiently well-defined that it is allowable to interchange the order of the integral and the sum. We therefore begin with the right-hand side of Eq. (1.6), intending to show that it indeed equals the left-hand side:

$$\frac{1}{L}\int_{-\frac{1}{2}L}^{+\frac{1}{2}L} dx\, f(x)\, e^{-ik_n x} = \sum_{m=-\infty}^{+\infty} c_m \frac{1}{L}\int_{-\frac{1}{2}L}^{+\frac{1}{2}L} dx\, e^{ik_m x} e^{-ik_n x} \ . \tag{1.7}$$

The value of the integral is given by

$$\frac{1}{L}\int_{-\frac{1}{2}L}^{+\frac{1}{2}L} dx\, e^{ik_m x} e^{-ik_n x} = \frac{1}{L}\int_{-\frac{1}{2}L}^{+\frac{1}{2}L} dx\, e^{i(2\pi/L)(m-n)x} = \frac{\sin[\pi(m-n)]}{\pi(m-n)} \ . \tag{1.8}$$

However, as both m and n are integers, this quantity is of course exactly zero, **except** for the special case when $m - n = 0$. In that case we must take the limit as $m \to n$, and, for instance, the power-series expansion of the sine function tells us that the limit is just $+1$, so that the value of the integral may be stated as δ_{mn}, the Kronecker delta, which has value $+1$ when its indices are equal, and is zero otherwise. The result of our calculation is then

$$\frac{1}{L}\int_{-\frac{1}{2}L}^{+\frac{1}{2}L} dx\, f(x)\, e^{-ik_n x} = \sum_{m=-\infty}^{+\infty} c_m \delta_{mn} = c_n \ . \tag{1.9}$$

This is in fact the value of the left-hand side, so that we have indeed proved consistency in this direction. This particular direction the result in question is often referred to by saying that the (basis) set $\{e^{(2\pi/L)nx}\}_{n=-\infty}^{+\infty}$ is *orthonormal,* as functions of x, over the interval of length L.

We now will show consistency in the other direction; as it turns out, this is somewhat more difficult, which is reasonable as, after all, it is the original proof of this direction that gave Fourier's name to the construction. We begin with the right-hand side of Eq. (1.5) and insert into it the value for c_n given by Eq. (1.6), but, again, noticing that the symbol x in Eq. (1.6) for c_n is just a "dummy" variable, i.e., a variable of integration, so that when we insert this value for c_n into Eq. (1.5) we must change the name of the variable of integration from x to something else, so we do not confuse it with the actual variable x that is on the left-hand side of the equation; we agree to call it y and then have the following, where, again, we assume that the functions involved are sufficiently well-defined that it is allowable to interchange the order of the integral and the sum:

$$\sum_{n=-\infty}^{+\infty} c_n \, e^{ik_n x} = \frac{1}{L} \int_{-\frac{1}{2}L}^{+\frac{1}{2}L} dy \; f(y) \sum_{n=-\infty}^{+\infty} e^{-ik_n y} e^{+ik_n x} \; . \tag{1.10}$$

Our problem then is to evaluate the sum above, which may also be written in a somewhat simpler form:

$$\sum_{n=-\infty}^{+\infty} e^{-ik_n y} e^{+ik_n x} = \sum_{n=-\infty}^{+\infty} e^{(2\pi i/L)n(x-y)} = 1 + 2\sum_{n=1}^{\infty} \cos[(2\pi/L)n(x-y)] \; . \tag{1.11}$$

It is clear that this sum is divergent when $x = y$. However, when this is not the case, it is not so clear what its value might be since the sign of its terms oscillate wildly as $|n|$ becomes larger and larger. To nonetheless determine a value we re-write it as follows:

$$\sum_{n=1}^{\infty} \cos[\alpha n(x-y)] = \frac{\partial}{\partial\alpha} \sum_{n=1}^{\infty} \frac{\sin[\alpha n(x-y)]}{\alpha} = \frac{\partial}{\partial\alpha} \frac{\pi - \alpha}{2} = -\frac{1}{2} \; , \tag{1.12}$$

where we have looked up the value of the last sum in a table of series. For instance, it is equation (1.44.1.1) in the book by Ryzhik and Gradshteyn, one of the best published tables of integrals and series. It gives the constraint on $\alpha(x-y)$ in order for the value to be valid, which is that $\alpha(x-y)$ must be larger than 0 and less than 2π, i.e., away from the boundaries of the period cell for the sine function. This obviously says that $x \neq y$; as well it says that $L > x - y$, which should be alright because we really only want the values of our periodic functions when their arguments range between $-L/2$ and $+L/2$.

Inserting this value into our equation we see that whenever $0 < x - y < L$, the value of our sum is simply zero! These two properties suggest to us that it should be proportional to a Dirac delta. We notice that Fourier's theorem would be true, using the definition of a Dirac delta, were it so that the sum is given by

$$\sum_{n=-\infty}^{+\infty} c_n \, e^{2\pi i(x-y)(n/L)} = L\delta(x-y) \; , \tag{1.13}$$

where we recall for the record the definition of the distribution named the Dirac delta. Denoting it by $\delta(x-y)$, so that it resembles the symbols for a function, and inserting it under an integral sign along with some other function, a so-called "test function," which is required to be at least reasonably nice, the definition is the following rule for the evaluation of the integral:

$$\int_a^b dy\, f(y)\,\delta(x-y) = \begin{cases} f(x)\,, & a < x < b\,, \\ 0\,, & \text{otherwise,} \end{cases} \tag{1.14}$$

where the "point" of the rule is that it vanishes everywhere **except** where its argument vanishes, at which point it evaluates the integral under which it has been placed. As an interesting, and probably useful, aside, we also note the following propositions about its behavior, which can be verified by changes of integration variables and/or integration by parts, where in all the cases below we would assume that the argument of the Dirac delta does indeed vanish during the region of integration, i.e., that both $+x$ and $-x$ lie within the interval between a and b:

$$\begin{aligned} \int_a^b dy\, f(y)\,\delta[\alpha(x-y)] &= \frac{1}{|\alpha|} f(x)\,, \\ \int_a^b dy\, f(y)\,\delta(x^2-y^2) &= \frac{f(x)+f(-x)}{2|x|}\,, \\ \int_a^b dy\, f(y)\,\frac{\partial}{\partial y}\delta(x-y) &= -\frac{\partial}{\partial x} f(x)\,. \end{aligned} \tag{1.15}$$

Returning to our Fourier series problem, and inserting this value for the sum into our integral, given above in Eq. (1.10), causes the desired result, namely that the integral has exactly the value $f(x)$, whenever $x-y$ lies between $-L/2$ and $+L/2$. As this is of course the content of Fourier's theorem, we presume that this evaluation is in fact correct—as indeed it is, provided both sides are interpreted as distributions, instead of just functions. In order to come up with a reasonable approach to a proof in terms of functions, we should consider not the sum from $n=-\infty$ to $+\infty$, but rather the sum from some $-N$ to $+N$, evaluate that sum, and then take the limit as $N \to \infty$. As an aside I note that the rigorous, mathematical reason this happens is that we really are not allowed to interchange the order of the integral and the sum, if we insist that all the quantities involved must be well-behaved functions. Lastly, one can note that there are a number of useful books outlining approaches to the theory of distributions, as "rules to evaluate integrals," and as limits of sequences of functions, which one might consult if desired:

II. Fourier Transforms: 1-dimensional

To eliminate the need to have periodic functions, i.e., so that we may take the approach above for a much larger class of functions, that are no longer required to be periodic, we take the limit as the period, L, approaches infinity. We will surely omit all the careful details concerning the requirements on the functions, $f(x)$, such that the limits all exist, but only give here an overall description of the process. We first take the sum in Eq. (1.5) and multiply it by +1, written in the form of the ratio of the two equal quantities, $2\pi/L$ and $\Delta k \equiv k_{n+1} - k_n = 2\pi/L$, which gives

$$f(x) = \sum_{n=-\infty}^{+\infty} \left(\frac{\Delta k}{2\pi/L}\right) c_n \, e^{ik_n x} = \sum_{n=-\infty}^{+\infty} \left(\frac{\Delta k}{2\pi}\right) [Lc_n] \, e^{ik_n x} \ . \tag{2.1}$$

In the limit as L becomes very large, heading toward infinity, clearly Δk becomes very small, heading toward the infinitesimal notion dk, while the sum over all values of n may be treated as an integral over dk, which still runs from $-\infty$ to $+\infty$, replacing n by its equivalent $kL/2\pi$, so that we may take k as the summation variable, that in the limit becomes an integration variable. This allows us to define the limit, as $L \to \infty$, of the (former) coefficients by a function of the (now continuous) variable k,namely $F(k)$, which is referred to as the *Fourier transform* of $f(x)$:

$$Lc_n = Lc_{_{kL/2\pi}} \quad \underset{L\to\infty}{\longrightarrow} \quad F(k) \ . \tag{2.2}$$

Using that limit, we may write the limit, as $L \to \infty$, of Eq. (1.5) for arbitrary, sufficiently-nice functions $f(x)$ in terms of their Fourier transforms:

$$f(x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \, F(k) \, e^{ikx} \ . \tag{2.3}$$

The **important physical statement about this is the following:** When considering $f(x)$ as a linear superposition of (infinitely) many harmonic waves, the role of $F(k)$ is to tell us how much and with what phase each individual monochromatic wave contributes to the entire superposition.

The inverse of this relationship will then allow for the determination of the Fourier transform when given the original function; this is of course the analog of Eq. (1.6) above, and should be taken easily in the limit of Eq. (1.6), after we multiply both sides by L, and take account of the limit of Lc_n as given in Eq. (2.2):

$$\lim_{L\to\infty} Lc_n = \lim_{L\to\infty} Lc_{_{kL/2\pi}} = F(k) = \int_{-\infty}^{+\infty} dx\, f(x)\, e^{-ikx} \ . \tag{2.4}$$

Again, surely, one should check the consistency of these two relationships. As it turns out insertion of one into the other and interchanging the orders of integration requires only one constraint, or definition, or rigorous mathematical proof, of the value of the following integral, which we have already discussed in class:

$$\int_{-\infty}^{+\infty} dk\, e^{ik(x-y)} = 2\pi\, \delta(x-y) \ . \tag{2.5}$$

III. Fourier Transforms: Some Useful Properties, and Examples

Let us think of the Fourier transform as a mapping of functions into other functions, also described as a mapping of a space of functions into itself: let $\mathcal{L}$ denote some appropriate space of functions, and let $\mathcal{F}$ denote this operation we are calling the Fourier transform; then we may describe this as

$$\mathcal{F} : \mathcal{L} \to \mathcal{L}\,, \text{ i.e., it is such that for every } f \in \mathcal{L}\ f = f(x), \quad \mathcal{F}(f) = F \in \mathcal{L}\,,\ F = F(k) \ . \tag{3.1}$$

Then this mapping has a number of useful properties, which we name below, and then give examples explaining each of them. In these examples, we will take g and h as functions of x, while α and β are just constant scalars:

1. Linearity: $f(x) = \alpha g(x) + \beta h(x) \quad \Longleftrightarrow \quad \mathcal{F}(f)(k) = \alpha\mathcal{F}(g)(k) + \beta\mathcal{F}(h)(k);$
2. Conjugation: $f(x) = g^*(x) \quad \Longleftrightarrow \quad \mathcal{F}(f)(k) = \{\mathcal{F}(g)\}^*(-k);$
3. the transform of the product of two functions is the convolution) of their transforms:

$$f(x) = g(x)h(x) \qquad \Longleftrightarrow \qquad \mathcal{F}(f)(k) = \int_{-\infty}^{+\infty} \frac{dk'}{2\pi}\, \mathcal{F}(g)(k')\mathcal{F}(h)(k-k') \ ;$$

4. the transform of a derivative:

$$f(x) = \frac{d^n}{dx^n} h(x) \qquad \Longleftrightarrow \qquad \mathcal{F}(f)(k) = (ik)^n \mathcal{F}(h)(k) \ ;$$

5. Parseval's relation for norms:

$$\int_{-\infty}^{+\infty} dx\, |f(x)|^2 = \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, |\mathcal{F}(f)(k)|^2 \ .$$

Some useful examples of particular Fourier transforms are given below:

$$\mathcal{F}(e^{-\gamma x} e^{i\kappa x}) = \frac{2\gamma}{(k-\kappa)^2 + \gamma^2} \ ,$$

$$\mathcal{F}(e^{-\gamma x^2} e^{i\kappa x}) = \sqrt{\frac{\pi}{\gamma}} e^{-(k-\kappa)^2/4\gamma} \ ,$$

$$\mathcal{F}(1) = 2\pi\, \delta(k) \ ,$$

$$\mathcal{F}(e^{isx}) = 2\pi\, \delta(s-k) \ ,$$

IV. Fourier Transforms in three dimensions:

The same statements may be made in three dimensions, with the "normative factor" 2π appearing once for each dimension:

$$f(\vec{r}) = \int_{(\infty)} \frac{d\vec{k}}{(2\pi)^3}\, F(\vec{k})\, e^{i\vec{k}\cdot\vec{r}} \quad \Longleftrightarrow \quad F(\vec{k}) = \int_{(\infty)} d\vec{r}\, f(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}} \ , \tag{4.1}$$

where the symbol (∞) under the integral sign is meant to indicate that one should integrate over the entire 3-dimensional space of the corresponding (vector-valued) variable of integration. In particular, then the truth of this statement requires the three-dimensional integral

$$\int_{(\infty)} d\vec{k}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')} = (2\pi)^3\, \delta^{(3)}(\vec{r}-\vec{r}\,') \ ,$$

or, equivalently,

$$\int_{-\infty}^{+\infty} dk_x \int_{-\infty}^{+\infty} dk_y \int_{-\infty}^{+\infty} dk_z\, e^{i[k_x(x-x')+k_y(y-y')+k_z(z-z')]} = (2\pi)^3\, \delta(x-x')\delta(y-y')\delta(z-z') \ .$$