The Lamb shift in H is not the end of the story.

In H the self-energy diagram dominates, vacuum-polarization is a small correction.

Muonic hydrogen is a proton + muon, μH.

For μH self-energy is small, V-P is large.

μH = p + μ, μ has charge -e

\[ m_\mu \approx 207 \text{ me} \]

so,
\[ a_{\mu} = 4\pi \alpha_0 \frac{\hbar^2}{m_\mu e^2} = \frac{me}{m_\mu} \alpha_0 \approx \frac{\alpha_0}{207} \]

reduced mass

The length scale of the vacuum polarization is

\[ \chi_c = \frac{\hbar}{m_\mu e} = \alpha \frac{\alpha_0}{m_\mu} \approx \frac{\alpha_0}{207} \]

small effect in H.

In μH, \[ \frac{\chi_c}{a_{\mu}} = \frac{\alpha \alpha_0}{a_{\mu}} = \frac{\alpha_0}{a_{\mu} m_\mu} \approx \frac{\alpha_0}{m_\mu} \approx 1.4 \]
Thus, V-P is very important.

To calculate the shift need field theory, out of scope for this course.

Nonetheless let's look at the scalings.


\[
\begin{align*}
2s_{\frac{1}{2}} - 2p_{\frac{1}{2}} & \quad \mu H \quad \Delta E_{\mu H} / \Delta E_H \\
\Delta E_{\text{self-energy}} & \quad +1.6 \text{ GHz} \quad +300.6 \text{ GHz} \quad \text{m\mu/m}\epsilon \\
\Delta E_{\text{vacuum-polarization}} & \quad -27 \text{ MHz} \quad -49.600 \text{ GHz} \quad (\text{m\mu/m}\epsilon)^3 \\
\end{align*}
\]

\[
\Delta E_{v-p} = \frac{2^3}{11 \pi} \frac{E_{\mu H}}{n^2} \left( \frac{-4}{15} S_{20} \right)
\]

\[
= \frac{-8}{15} \frac{2^3}{n^3} E_{\mu H} S_{20} + \text{higher order terms}
\]

\[
E_{\mu H} \sim 207 E_H
\]

The \( S_{20} \) term picks out s-states, as in the Darwin term calculation.
Recall

\[ |\psi(0)|^2 = \frac{1}{\pi a_0^3 n^3} \sim \frac{m_e}{m_u} \]

\[ \text{in } \mu H \quad |\psi(0)|^2 \sim \frac{m_e}{m_u} \]

So we get a larger effect \( \sim \left( \frac{m_u}{m_e} \right)^3 \sim 10^7 \)

The Lamb shift in \( \mu H \) was measured,

R. Pohl, et al. "The size of the proton"

*Nature, 466, 213 (2010)*.
So why was the experiment measurement of the Lamb shift in μHz called "size of the proton"?

Theory says that, if we include proton size effects, then

$$\Delta E_{\text{Lamb}} = 209 - 5.2 \, r_p^2 + 0.03 \, r_p^3 + \ldots \text{meV}$$

$$r_p = \sqrt{2r_p^2}$$ is the charge radius of the proton.

The 2010 experiment implied a value for $r_p$ that disagrees by 566 from the 2006 CODATA value.

Work is ongoing ---