Angular momentum in Quantum Mechanics

Detailed understanding of the structure of atoms relies on the quantum mechanical description of angular momentum. In this section we summarize some essential parts of the quantum theory of angular momentum. The description is brief and is not intended to be a substitute for a quantum mechanics textbook which should be consulted for proofs of the results given below[1]

1.1 Angular momentum operators and states

We denote the angular momentum operator by $\hat{\mathbf{J}}$ which has Cartesian components[2] $\hat{\mathbf{J}} = \hat{J}_x \mathbf{e}_x + \hat{J}_y \mathbf{e}_y + \hat{J}_z \mathbf{e}_z$. Any operator satisfying the commutation relations $[\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk}\hbar \hat{J}_k$ is an angular momentum. Equivalently any operator satisfying $\hat{\mathbf{J}} \times \hat{\mathbf{J}} = i\hbar\hat{\mathbf{J}}$ is an angular momentum. A complete set of commuting observables (CSCO) for states carrying angular momentum is provided by the operators $\hat{J}^2$, $\hat{J}_z$. The eigenstates of these operators are labeled $|j,m\rangle$ where $j \geq 0$ is the angular momentum and m is the "magnetic" quantum number which gives the projection of $\hat{\mathbf{J}}$ on the quantization axis which we will take to be $\mathbf{e}_z$. The possible values of these quantities are $j \geq 0$, integer or half-integer values only being allowed and $-j \leq m \leq j$, with successive values of m being separated by one. For example for $j = 1$ we have $m = -1, 0, 1$ and for $j = 3/2$, $m = -3/2, -1/2, 1/2, 3/2$.

The states are orthogonal, $\langle j', m'|j, m\rangle = \delta_{jj'}\delta_{mm'}$ and are complete $\sum_{m=-j}^{j} |j,m\rangle\langle j,m| = \hat{I}$, where the identity operator acts on a subspace with the given value of j. The eigenvalue relations are

$$\hat{J}^2|j,m\rangle = j(j+1)\hbar^2|j,m\rangle \tag{1.1a}$$

$$\hat{J}_z|j,m\rangle = m\hbar|j,m\rangle. \tag{1.1b}$$

We characterize the magnitude of the angular momentum by $\sqrt{\langle \hat{J}^2\rangle} = \sqrt{j(j+1)}$, which is greater than the "value" of the angular momentum j.

It is convenient to introduce the raising and lowering operators[3]

$$\hat{J}_\pm = \mp\frac{1}{\sqrt{2}}\left(\hat{J}_x \pm i\hat{J}_y\right). \tag{1.2}$$

The square of the angular momentum can be written in terms of these operators as $\hat{J}^2 = -\hat{J}_+\hat{J}_- - \hat{J}_-\hat{J}_+ + \hat{J}_z^2$. These operators raise or lower the value of m according to

$$\hat{J}_\pm|j,m\rangle = \mp\frac{1}{\sqrt{2}}\sqrt{j(j+1) - m(m\pm 1)}\hbar|j, m\pm 1\rangle. \tag{1.3}$$

[1] Good references for angular momentum theory include M. Weissbluth, "Atoms and molecules", student edition, (Academic, New York, 1978), D. A. Varshalovich, A. N. Moskalev, and V. K. Khersonskii, "Quantum theory of angular momentum", World Scientific, Singapore, 1989). A word of caution: many different conventions are in use regarding minus signs and where to put factors of 2, etc. These notes do not coincide completely with the notation of Basdevant & Dalibard.

[2] Since hats denote quantum operators we use $\mathbf{e}_x$ etc., instead of the more customary $\hat{x}$ to denote unit vectors.

[3] These definitions are slightly different than those in Basdevant & Dalibard in order to be compatible with the definitions of spherical basis vectors used in Sec. 1.3.

We also have the commutation relations $[\hat{J}^2, \hat{J}_\pm] = 0$, $[\hat{J}_z, \hat{J}_\pm] = \pm\hbar\hat{J}_\pm$, and $[\hat{J}_+, \hat{J}_-] = -\hbar\hat{J}_z$.

In the case of integer values of $j = l$ the angular momentum states can be written in a coordinate representation in terms of the spherical harmonics $Y_{l,m}(\theta, \phi)$. The eigenvalue relations then take the form

$$\hat{L}^2 Y_{l,m}(\theta,\phi) = l(l+1)\hbar^2 Y_{l,m}(\theta,\phi) \tag{1.4a}$$
$$\hat{L}_z Y_{l,m}(\theta,\phi) = m\hbar Y_{l,m}(\theta,\phi) \tag{1.4b}$$

where

$$\hat{L}^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) \tag{1.5a}$$
$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}. \tag{1.5b}$$

1.1.1 Addition of angular momenta

Two angular momenta $\hat{\mathbf{J}}_1, \hat{\mathbf{J}}_2$ can be combined to give a coupled angular momentum

$$\hat{\mathbf{J}} = \hat{\mathbf{J}}_1 + \hat{\mathbf{J}}_2 = \hat{J}_x\mathbf{e}_x + \hat{J}_y\mathbf{e}_y + \hat{J}_z\mathbf{e}_z = (\hat{J}_{1x} + \hat{J}_{2x})\mathbf{e}_x + (\hat{J}_{1y} + \hat{J}_{2y})\mathbf{e}_y + (\hat{J}_{1z} + \hat{J}_{2z})\mathbf{e}_z. \tag{1.6}$$

We assume that the angular momenta $\hat{\mathbf{J}}_1, \hat{\mathbf{J}}_2$ act in different spaces and commute with each other. For example we could have $\hat{\mathbf{J}}_1 = \hat{\mathbf{L}}$ and $\hat{\mathbf{J}}_2 = \hat{\mathbf{s}}$ corresponding to orbital and spin degrees of freedom of a particle. Alternatively we could have $\hat{\mathbf{J}}_1 = \hat{\mathbf{s}}_1$ and $\hat{\mathbf{J}}_2 = \hat{\mathbf{s}}_2$ corresponding to spin degrees of freedom of two different particles.

Consider an uncoupled basis corresponding to the observables $\hat{\mathbf{J}}_1, \hat{\mathbf{J}}_2$. A CSCO is formed by the operators $\left\{\hat{J}_1^2, \hat{J}_{1z}, \hat{J}_2^2, \hat{J}_{2z}\right\}$. The states are labelled $|j_1, m_1; j_2; m_2\rangle$ and the eigenvalue relations are

$$\hat{J}_1^2|j_1, m_1; j_2; m_2\rangle = j_1(j_1+1)\hbar^2|j_1, m_1; j_2; m_2\rangle \tag{1.7a}$$
$$\hat{J}_{1z}|j_1, m_1; j_2; m_2\rangle = m_1\hbar|j_1, m_1; j_2; m_2\rangle \tag{1.7b}$$
$$\hat{J}_2^2|j_1, m_1; j_2; m_2\rangle = j_2(j_2+1)\hbar^2|j_1, m_1; j_2; m_2\rangle \tag{1.7c}$$
$$\hat{J}_{2z}|j_1, m_1; j_2; m_2\rangle = m_2\hbar|j_1, m_1; j_2; m_2\rangle. \tag{1.7d}$$

There are $(2j_1+1)(2j_2+1)$ states in this uncoupled basis.

We can couple the eigenstates to form states $|j_1, j_2; j, m\rangle$ where $|j_1 - j_2| \le j \le j_1 + j_2$. In the coupled basis a CSCO is given by the operators $\left\{\hat{J}_1^2, \hat{J}_2^2, \hat{J}^2, \hat{J}_z\right\}$. The eigenvalue relations in the coupled basis are

$$\hat{J}^2|j_1, j_2; j, m\rangle = j(j+1)\hbar^2|j_1, j_2; j, m\rangle \tag{1.8a}$$
$$\hat{J}_z|j_1, j_2; j, m\rangle = m\hbar|j_1, j_2; j, m\rangle \tag{1.8b}$$
$$\hat{J}_1^2|j_1, j_2; j, m\rangle = j_1(j_1+1)\hbar^2|j_1, j_2; j, m\rangle \tag{1.8c}$$
$$\hat{J}_2^2|j_1, j_2; j, m\rangle = j_2(j_2+1)\hbar^2|j_1, j_2; j, m\rangle. \tag{1.8d}$$

In the coupled basis j takes on the possible values $|j_1 - j_2|, |j_1 - j_2| + 1, ...j_1 + j_2$ and $-j \le m \le j$. The total number of states is unchanged since $\sum_{j=|j_1-j_2|}^{j_1+j_2} 2j + 1 = (2j_1+1)(2j_2+1)$.

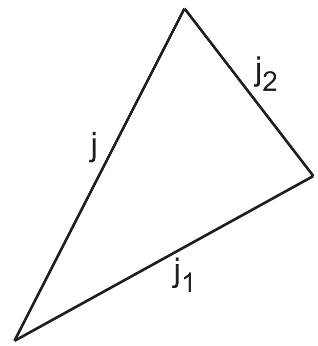

Figure 1.1: Triangle condition for addition of angular momenta.

The transformation from the uncoupled to the coupled bases is expressed through the Clebsch-Gordan coefficients. They can be written as

$$\begin{aligned}
|j_1, j_2; j, m\rangle &= \hat{I}|j_1, j_2; j, m\rangle \\
&= \left(\sum_{m_1,m_2} |j_1, m_1; j_2; m_2\rangle\langle j_1, m_1; j_2; m_2|\right)|j_1, j_2; j, m\rangle \\
&= \sum_{m_1,m_2} \langle j_1, m_1; j_2; m_2|j_1, j_2; j, m\rangle|j_1, m_1; j_2; m_2\rangle \\
&= \sum_{m_1,m_2} C^{jm}_{j_1m_1j_2m_2}|j_1, m_1; j_2; m_2\rangle, \qquad (1.9)
\end{aligned}$$

where

$$C^{jm}_{j_1m_1j_2m_2} = \langle j_1, m_1; j_2; m_2|j_1, j_2; j, m\rangle \qquad (1.10)$$

are the Clebsch-Gordan coefficients. These coefficients vanish unless

$$|j_1 - j_2| \leq \ j \ \leq j_1 + j_2 \qquad (1.11a)$$

$$m_1 + m_2 = m. \qquad (1.11b)$$

The first of these conditions is known as the triangle condition, and the second condition expresses the conservation of angular momentum along the quantization axis.

The triangle condition can be understood in terms of vector additions as shown in Fig. 1.1. Since any two of the angular momenta can be combined to give the third we require that

$$j \leq j_1 + j_2 \qquad (1.12a)$$

$$j_1 \leq j_2 + j \qquad (1.12b)$$

$$j_2 \leq j_1 + j. \qquad (1.12c)$$

The last two conditions can be written as $j \geq j_1 - j_2$ and $j \geq j_2 - j_1$ which can be combined to give $j \geq |j_1 - j_2|$. This together with the first condition gives (1.11a).

There are many different notations in use for the Clebsch-Gordan coefficients. Many authors prefer to use the $3j$ symbols $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ which have a higher symmetry. The

$3j$ symbols can be defined by

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix} = \frac{(-1)^{j_1-j_2+m}}{\sqrt{2j+1}} C^{jm}_{j_1m_1j_2m_2}. \tag{1.13}$$

1.2 Scalar and vector operators

Quantum mechanical operators can be grouped according to their properties under rotation. The simplest case is that of a scalar operator which is invariant under rotations. This is expressed mathematically by the statement that a scalar operator $\hat{O}$ commutes with all components of the angular momentum $\hat{\mathbf{J}}$, i.e. $[\hat{O}, \hat{\mathbf{J}}] = 0$.

To see that this implies that expectation values are independent of rotation of the operator consider the matrix elements $\langle \alpha, j, m|\hat{O}|\beta, j', m'\rangle$. Here α, β are additional quantum numbers which do not depend on the orientation of the states. There are a total of $(2j+1)(2j'+1)$ matrix elements. We want to show that if $j \neq j'$ or $m \neq m'$ the matrix elements vanish, and when $j = j'$ and $m = m'$ all $2j+1$ matrix elements are equal.

To prove this consider the commutator $[\hat{O}, \hat{J}_z] = 0$. Thus

$$0 = \langle \alpha, j, m|[\hat{O}, \hat{J}_z]|\beta, j', m'\rangle = (m'-m)\hbar\langle \alpha, j, m|\hat{O}|\beta, j', m'\rangle.$$

So if $m' \neq m$ the matrix element must vanish and if $m' = m$ we define the quantity $O_m = \langle \alpha, j, m|\hat{O}|\beta, j', m\rangle$.

We then consider the commutators $[\hat{J}_\pm, \hat{O}] = 0$. evaluation of matrix elements of these commutators leads to the equalities

$$\sqrt{j(j+1)-m(m+1)}O_m = \sqrt{j'(j'+1)-m(m+1)}O_{m+1} \tag{1.14a}$$
$$\sqrt{j(j+1)-m(m+1)}O_{m+1} = \sqrt{j'(j'+1)-m(m+1)}O_m. \tag{1.14b}$$

If $j \neq j'$ there is a contradiction unless $O_m = 0$ for all m and if $j = j'$ all the O_m are equal. Thus for any scalar operator $\hat{O}$ we have that

$$\langle \alpha, j, m|\hat{O}|\beta, j', m'\rangle = O_m \delta_{jj'}\delta_{mm'}.$$

This is a convenient result since it is enough to calculate a single matrix element and the value of m can be chosen to simplify the calculation as much as possible. Examples of scalar operators are $\hat{\mathbf{r}}^2, \hat{\mathbf{p}}^2, \hat{\mathbf{J}}^2$.

One might think that any scalar quantity is a scalar operator, but this is not the case. Consider the Zeeman Hamiltonian $\hat{H} = -\hat{\boldsymbol{\mu}} \cdot \mathbf{B}$. $\hat{H}$ is a scalar, but if it were a scalar operator levels with different values of m would have the same Zeeman shift when a magnetic field is applied. This disagrees with experiment. The explanation is that a scalar operator is invariant under rotations of the operator with respect to the quantization axis. When the atom is rotated the direction of the dipole moment changes, but the magnetic field stays fixed which breaks the spherical symmetry.

The next type of operator to consider is a vector operator which we define by requiring that expectation values transform under rotation in the same way as classical vectors do. In classical physics a vector $\mathbf{V} = V_x\mathbf{e}_x + V_y\mathbf{e}_y + V_z\mathbf{e}_z$ transforms to $\mathbf{V}' = \mathbf{R}\mathbf{V} = V'_x\mathbf{e}_x + V'_y\mathbf{e}_y + V'_z\mathbf{e}_z$

where $\mathbf{R}$ is a 3×3 orthogonal rotation matrix. In terms of Cartesian components this can be written as $V_i' = \sum_j R_{ij} V_j$.

In quantum mechanics we require correspondingly that the expectation value of each component of $\hat{\mathbf{V}}$ transforms as

$$\langle\psi|\hat{\mathbf{V}}|\psi\rangle_i = \langle\psi|\hat{V}_i|\psi\rangle \rightarrow \langle\psi|\sum_j \hat{R}_{ij}\hat{V}_j|\psi\rangle = \sum_j \hat{R}_{ij}\langle\psi|\hat{V}_j|\psi\rangle. \tag{1.15}$$

A quantum mechanical state $|\psi\rangle$ transforms as

$$|\psi\rangle \rightarrow \hat{D}_{\mathbf{R}}|\psi\rangle \quad \text{and} \quad \langle\psi| \rightarrow \langle\psi|\hat{D}_{\mathbf{R}}^\dagger$$

where $\hat{D}_{\mathbf{R}}$ is a rotation operator[4]. Thus

$$\langle\psi|\hat{V}_i|\psi\rangle \rightarrow \langle\psi|\hat{D}_{\mathbf{R}}^\dagger\hat{V}_i\hat{D}_{\mathbf{R}}|\psi\rangle \tag{1.16}$$

and the requirement that the right hand sides of (1.15,1.16) are equal for arbitrary $|\psi\rangle$ results in

$$\hat{D}_{\mathbf{R}}^\dagger\hat{V}_i\hat{D}_{\mathbf{R}} = \sum_j \hat{R}_{ij}\hat{V}_j \ . \tag{1.17}$$

Consider the limit of an infinitesimal rotation by an angle $\delta\theta$ about the $\mathbf{e}_z$ axis. Then

$$\hat{D}_{\mathbf{R}} = 1 - i\frac{\delta\theta\hat{J}_z}{\hbar}$$

and

$$\mathbf{R} = \begin{pmatrix} 1 & -\delta\theta & 0 \\ \delta\theta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Keeping only linear terms in (1.17) leads to

$$[\hat{V}_x, \hat{J}_z] = i\hbar\hat{V}_y, \quad [\hat{V}_y, \hat{J}_z] = -i\hbar\hat{V}_x, \quad [\hat{V}_z, \hat{J}_z] = 0.$$

If we repeat the calculation for rotation about the other Cartesian axes we can write the results as a single commutation relation

$$[\hat{V}_i, \hat{J}_j] = i\epsilon_{ijk}\hbar\hat{V}_k. \tag{1.18}$$

Any operator which satisfies Eq. (1.18) thus has expectation values which transform under infinitesimal rotations in the same way as ordinary vectors do. Since finite rotations can be described as a succession of infinitesimal rotations the same relationship must hold. We therefore take Eq. (1.18) as the defining property of a vector operator. Examples of vector operators are the position $\hat{\mathbf{r}}$, momentum $\hat{\mathbf{p}}$, and orbital angular momentum $\hat{\mathbf{L}}$ of a point particle.

[4]The operator for rotation of a state about axis $\mathbf{e}_j$ by an angle θ is $\hat{D}_{\mathbf{R}} = e^{-\imath\theta\hat{J}_j/\hbar}$, see HW set 5 from PH448. The case worked out there was for orbital angular momentum but the result generalizes to an arbitrary angular momentum $\hat{\mathbf{J}}$. An arbitrary rotation in three dimensions can be described by successive application of rotation operators about diferent axes.

1.3 Spherical coordinates

Why do we care whether or not an operator is a vector operator? The usefulness of this identification appears when we wish to calculate matrix elements of vector operators, which are greatly simplified by use of the Wigner-Eckart theorem. However, in order to apply this theorem in a convenient form it is necessary to first work in a spherical basis instead of the Cartesian coordinate system we have been using so far.

An arbitrary vector $\mathbf{A}$ can be written in terms of Cartesian unit vectors as

$$\mathbf{A} = A_x \mathbf{e}_x + A_y \mathbf{e}_y + A_z \mathbf{e}_z.$$

Alternatively we can use spherical basis vectors $\{\mathbf{e}_+, \mathbf{e}_0, \mathbf{e}_-\}$ which are defined by

$$\begin{aligned} \mathbf{e}_0 &= \mathbf{e}_z, \quad \mathbf{e}_1 = \mathbf{e}_+ = \frac{-1}{\sqrt{2}}(\mathbf{e}_x + i\mathbf{e}_y), \quad \mathbf{e}_{-1} = \mathbf{e}_- = \frac{1}{\sqrt{2}}(\mathbf{e}_x - i\mathbf{e}_y) \\ \mathbf{e}_0^* &= \mathbf{e}_z, \quad \mathbf{e}_1^* = \mathbf{e}_+^* = \frac{-1}{\sqrt{2}}(\mathbf{e}_x - i\mathbf{e}_y), \quad \mathbf{e}_{-1}^* = \mathbf{e}_-^* = \frac{1}{\sqrt{2}}(\mathbf{e}_x + i\mathbf{e}_y) \end{aligned}$$

We see that $\mathbf{e}_p^* = (-1)^p \mathbf{e}_{-p}$ and

$$\mathbf{e}_p^* \mathbf{e}_q = (-1)^p \mathbf{e}_{-p} \mathbf{e}_q = \delta_{pq}.$$

The inverse transformations are

$$\mathbf{e}_x = \frac{-1}{\sqrt{2}}(\mathbf{e}_1 - \mathbf{e}_{-1}), \quad \mathbf{e}_y = \frac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_{-1}), \quad \mathbf{e}_z = \mathbf{e}_0.$$

The q^{th} component of a vector $\mathbf{A}$ is by definition $A_q = \mathbf{A} \cdot \mathbf{e}_q$. We wish to write $\mathbf{A}$ such that this definition holds in the spherical basis, and therefore write $\mathbf{A} = \sum_q A_q \tilde{\mathbf{e}}_q$, where the basis vectors $\tilde{\mathbf{e}}_q$ are to be determined. We therefore require

$$\mathbf{A} \cdot \mathbf{e}_q = \sum_{q'} A_{q'} \tilde{\mathbf{e}}_{q'} \cdot \mathbf{e}_q = A_q.$$

This is true provided $\tilde{\mathbf{e}}_{q'} \cdot \mathbf{e}_q = \delta_{q'q}$ which implies $\tilde{\mathbf{e}}_q = \mathbf{e}_q^*$. Thus in the spherical basis we have

$$\mathbf{A} = \sum_{q=-1,0,1} A_q \mathbf{e}_q^*$$

where

$$A_0 = A_z, \quad A_1 = A_+ = \frac{-1}{\sqrt{2}}(A_x + iA_y), \quad A_{-1} = A_- = \frac{1}{\sqrt{2}}(A_x - iA_y). \tag{1.19}$$

We note that an arbitrary vector $\mathbf{A}$ can equivalently be written as

$$\mathbf{A} = \sum_{q=-1,0,1} (-1)^q A_{-q}^* \mathbf{e}_q,$$

where the A_q are given by Eq. (1.19).

The dot product of two real vectors is

$$\begin{aligned}
\mathbf{A}\cdot\mathbf{B} &= \sum_{j=x,y,z} A_j B_j \\
&= \left(\sum_{p=-1,0,1} A_p \mathbf{e}_p^*\right)\cdot\left(\sum_{q=-1,0,1} B_q \mathbf{e}_q^*\right) \\
&= \sum_{p,q=-1,0,1} A_p B_q \mathbf{e}_p^*\cdot\mathbf{e}_q^* \\
&= \sum_{p,q=-1,0,1} A_p B_q \mathbf{e}_p^*\cdot(-1)^q\mathbf{e}_{-q} \\
&= \sum_{p,q=-1,0,1} A_p B_q (-1)^q \delta_{p,-q} \\
&= \sum_{p=-1,0,1} A_p B_{-p}(-1)^p \\
&= \sum_{p=-1,0,1} A_p B_p^* = \sum_{p=-1,0,1} A_p^* B_p. \qquad (1.20)
\end{aligned}$$

We can express any vector $\mathbf{A}$ in terms of the spherical polar angles θ,ϕ as

$$\begin{aligned}
A_0(\theta,\phi) &= |\mathbf{A}|\cos\theta = |\mathbf{A}|\sqrt{\frac{4\pi}{3}}Y_{10}(\theta,\phi) \\
A_1(\theta,\phi) &= -|\mathbf{A}|\frac{e^{\imath\phi}\sin\theta}{\sqrt{2}} = |\mathbf{A}|\sqrt{\frac{4\pi}{3}}Y_{11}(\theta,\phi) \\
A_{-1}(\theta,\phi) &= |\mathbf{A}|\frac{e^{-\imath\phi}\sin\theta}{\sqrt{2}} = |\mathbf{A}|\sqrt{\frac{4\pi}{3}}Y_{1-1}(\theta,\phi). \qquad (1.21)
\end{aligned}$$

Thus

$$\mathbf{A}(\theta,\phi) = \sum_{q=-1,0,1} A_q(\theta,\phi)\mathbf{e}_q^* = |\mathbf{A}|\sqrt{\frac{4\pi}{3}}\sum_q Y_{1q}(\theta,\phi)\mathbf{e}_q^*. \qquad (1.22)$$

Note that the raising and lowering operators for the angular momentum defined in Eq. (1.2) are simply proportional to the $\pm$ components of the angular momentum in a spherical basis since

$$\hat{\mathbf{J}} = \hat{J}_x\mathbf{e}_x + \hat{J}_y\mathbf{e}_y + \hat{J}_z\mathbf{e}_z = -\hat{J}_+\mathbf{e}_- - \hat{J}_-\mathbf{e}_+ + \hat{J}_0\mathbf{e}_0.$$

Relation (1.18) which we are using to define a vector operator can be written in a spherical basis as

$$[\hat{J}_0,\hat{V}_q] = \hbar q\hat{V}_q, \qquad (1.23a)$$

$$\left[\hat{J}_\pm,\hat{V}_q\right] = \mp\hbar\frac{1}{\sqrt{2}}\sqrt{2-q(q\pm1)}\hat{V}_{q\pm1}, \qquad (1.23b)$$

where $q = 1, 0, -1$.

1.4 Spherical tensors

Tensors can be thought of as higher dimensional generalizations of scalars and vectors. In classical physics tensors transform according to

$$T_{ijk...} \rightarrow T'_{ijk...} = \sum_r \sum_s \sum_t ... \, R_{ir} R_{js} R_{kt} ... \, T_{rst...} \quad . \tag{1.24}$$

The number of indices of the tensor T defines its rank. A scalar is a rank 0 tensor, and a vector is a rank 1 tensor. In quantum mechanics we will mostly be only interested in tensors of rank 0,1, or 2. A Cartesian tensor of rank 2 can be formed from two vectors $\mathbf{U}$, $\mathbf{V}$ giving what is called a dyadic:

$$T_{ij} = U_i V_j. \tag{1.25}$$

This rank 2 tensor has $3 \times 3 = 9$ components since $i = x, y, z$ and $j = x, y, z$ and it transforms according to Eq. (1.24) where two rotation matrices appear.

It turns out that it is not convenient to work with Cartesian tensors because they are reducible. That is to say a Cartesian tensor can be decomposed into objects which each transform like tensors of different ranks. Consider the rank 2 Cartesian tensor defined above. We can equivalently write it as

$$T_{ij} = \left(\frac{\mathbf{U} \cdot \mathbf{V}}{3} \delta_{ij} \right) + \left(\frac{U_i V_j - U_j V_i}{2} \right) + \left(\frac{U_i V_j + U_j V_i}{2} - \frac{\mathbf{U} \cdot \mathbf{V}}{3} \delta_{ij} \right). \tag{1.26}$$

It can be readily checked that the components T_{ij} defined by Eqs. (1.25) or (1.26) are the same. However the three terms in parentheses which appear in Eq. (1.26) have very different transformation properties. They are irreducible Cartesian tensors of ranks $0, 1$, and 2. We can write

$$T_{ij} = T_{ij}^{(0)} + T_{ij}^{(1)} + T_{ij}^{(2)} \tag{1.27}$$

where

$$T_{ij}^{(0)} = \frac{\mathbf{U} \cdot \mathbf{V}}{3} \delta_{ij} \tag{1.28a}$$

$$T_{ij}^{(1)} = \frac{U_i V_j - U_j V_i}{2} \tag{1.28b}$$

$$T_{ij}^{(2)} = \frac{U_i V_j + U_j V_i}{2} - \frac{\mathbf{U} \cdot \mathbf{V}}{3} \delta_{ij} \tag{1.28c}$$

The first term $T_{ij}^{(0)}$ is a scalar which is invariant under rotation. The second term $T_{ij}^{(1)}$ is an antisymmetric tensor with zero trace (sum of the diagonal elements) which can be written as the vector product $\epsilon_{ijk} (\mathbf{U} \times \mathbf{V})_k / 2$ and transforms as a vector. The last term $T_{ij}^{(2)}$ is a symmetric rank 2 tensor with zero trace which transforms as a tensor.

The number of independent components of these three terms is 1 (for the scalar), 3 (for the vector), and 5 (for the tensor). The total number of independent components is unchanged since

$$3 \times 3 = 9 = 1 + 3 + 5.$$

The numbers on the right hand side correspond to the number of possible states for objects with angular momenta $0, 1$, and 2. Indeed by writing T_{ij} in the form of Eq. (1.27) we have decomposed it into irreducible spherical tensors, which are objects that transform like angular momentum states (or spherical harmonics) with $l = 0, 1, 2$.

The decomposition (1.27) is the irreducible representation of the Cartesian tensor T_{ij}. We write spherical tensors of rank $0, 1, 2$ as $\mathbf{T}_0, \mathbf{T}_1$, and $\mathbf{T}_2$. It can be shown that the elements $T_{\kappa,q}$ of the spherical tensors of rank κ are related to the Cartesian components of the rank 2 tensor T_{ij} by

$$\begin{aligned}
T_{0,0} &= \frac{1}{3}\sum_i T_{ii} \\
T_{1,0} &= T^{(1)}_{xy} \\
T_{1,\pm1} &= \mp\frac{1}{\sqrt{2}}\left(T^{(1)}_{yz} \pm iT^{(1)}_{zx}\right) \\
T_{2,0} &= T^{(2)}_{zz} \\
T_{2,\pm1} &= \mp\sqrt{\frac{2}{3}}\left(T^{(2)}_{zx} \pm iT^{(2)}_{zy}\right) \\
T_{2,\pm2} &= \sqrt{\frac{1}{6}}\left[T^{(2)}_{xx} - T^{(2)}_{yy} \pm 2iT^{(2)}_{xy}\right].
\end{aligned}$$

These relations are specific to the decomposition of a rank 2 tensor. Higher rank Cartesian tensors are more complicated to work with since their irreducible representation is not unique. We define a spherical tensor of rank κ as an operator which satisfies the commutation relations

$$[\hat{J}_0, \hat{T}_{\kappa,q}] = \hbar q \hat{T}_{\kappa,q}, \tag{1.29a}$$

$$\left[\hat{J}_\pm, \hat{T}_{\kappa,q}\right] = \mp\hbar\frac{1}{\sqrt{2}}\sqrt{\kappa(\kappa+1) - q(q\pm1)}\hat{T}_{\kappa,q\pm1}, \tag{1.29b}$$

where $-\kappa \le q \le \kappa$. A vector is a first rank tensor and putting $\kappa = 1$ in this definition recovers the commutation relations for vector operators given in Eqs. (1.23).

1.5 Matrix elements of spherical tensors and the Wigner-Eckart theorem

After these preliminaries we come to an important result which is how to calculate matrix elements using the Wigner-Eckart theorem. Before stating the general form of this theorem let's consider the example of calculating matrix elements of the position operator between angular momentum states. This task is important for example when calculating dipole matrix elements and Rabi frequencies for optically induced transitions between atomic states.

The matrix element of the position operator between angular momentum states is

$$\begin{aligned}
\langle n'l'm'|\hat{\mathbf{r}}|nlm\rangle &= \langle n'l'm'|r\sqrt{\frac{4\pi}{3}}\sum_q Y_{1q}|nlm\rangle \mathbf{e}^*_q \\
&= \langle n'l'|r|nl\rangle\sqrt{\frac{4\pi}{3}}\sum_q \mathbf{e}^*_q\langle l'm'|Y_{1q}|lm\rangle.
\end{aligned}$$

Here n, n' are radial or other quantum numbers specifying degrees of freedom which do not depend on the angular coordinates. To find the angular matrix element we use the identity

$$Y_{l_1m_1}Y_{l_2m_2} = \sum_{lm}\left[\frac{(2l_1+1)(2l_2+1)(2l+1)}{4\pi}\right]^{1/2}\begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m\end{pmatrix} Y^*_{lm}\begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m\end{pmatrix}$$

where the terms in parentheses are Wigner 3j symbols. The symbol

$$\begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m\end{pmatrix}$$

can be nonzero when $m_1+m_2+m=0$ and l satisfies the triangle inequality $|l_1-l_2| \le l \le l_1+l_2$ so that l takes one of the values $l = l_1+l_2, l_1+l_2-1, ..., |l_1-l_2|$.

Thus

$$\langle l'm'|Y_{1q}|lm\rangle = \int d\Omega\, Y^*_{l'm'}Y_{1q}Y_{lm} = \sqrt{\frac{3}{4\pi}}\sqrt{(2l+1)(2l'+1)}(-1)^{m'}\begin{pmatrix} l' & 1 & l \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l' & 1 & l \\ -m' & q & m\end{pmatrix} \tag{1.30}$$

and we get

$$\langle n'l'm'|\hat{\mathbf{r}}|nlm\rangle = \langle n'l'|r|nl\rangle\sqrt{(2l+1)(2l'+1)}(-1)^{m'}\sum_q \mathbf{e}^*_q\begin{pmatrix} l' & 1 & l \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l' & 1 & l \\ -m' & q & m\end{pmatrix} \tag{1.31}$$

which is known as the Gaunt formula. The matrix element of an individual component is nonzero only when $l' = l \pm 1$ and $m' = m + q$. We note that the matrix elements are the product of a factor $\langle n'l'|r|nl\rangle$ which is independent of m, m', q and a geometrical factor which does depend on m, m', q. Thus, the dependence on orientation of the matrix elements is always the same, for any radial dependence of the wavefunctions.

1.5.1 Wigner-Eckart theorem

The Gaunt formula gives a complete description for the matrix elements of the position operator between angular momentum states $|lm\rangle$. The result can be generalized to matrix elements of arbitrary spherical tensors using the Wigner-Eckart theorem. It states that

$$\langle \alpha j'm'|\hat{T}_{\kappa q}|\beta jm\rangle = \frac{\langle \alpha j'||\hat{\mathbf{T}}_\kappa||\beta j\rangle}{\sqrt{2j'+1}}(-1)^{2\kappa}C^{j'm'}_{jm\kappa q} \tag{1.32}$$

Here $\hat{T}_{\kappa q}$ is the $q^{\rm th}$ component of a spherical tensor operator $\hat{\mathbf{T}}_\kappa$ of rank κ with $2\kappa+1$ components $-\kappa \le q \le \kappa$. The tensor components satisfy the angular momentum commutation relations given in Eqs. (1.29). The symbol $\langle \alpha j'||\hat{\mathbf{T}}_\kappa||\beta j\rangle$ is known as a reduced matrix element which is independent of m, m'. For completeness we note that the Wigner-Eckart theorem can also be written in terms of 3j symbols as

$$\langle \alpha j'm'|\hat{T}_{\kappa q}|\beta jm\rangle = (-1)^{j'-m'}\langle \alpha j'||\hat{\mathbf{T}}_\kappa||\beta j\rangle\begin{pmatrix} j' & \kappa & j \\ -m' & q & m\end{pmatrix}. \tag{1.33}$$

Using Eq. (1.32) and the properties of the Clebsch-Gordan coefficients we can immediately write down selection rules for transition matrix elements of spherical tensor operators. For scalar operators ($\kappa = 0, q = 0$)

$$j' = j \tag{1.34a}$$
$$m' = m, \tag{1.34b}$$

for vector operators ($\kappa = 1, q = -1, 0, 1$)

$$j' = j-1, j, j+1 \tag{1.35a}$$
$$j + j' \geq \kappa\ (= 1) \tag{1.35b}$$
$$m' = m-1, m, m+1, \tag{1.35c}$$

and for rank 2 tensor operators

$$j' = j-2, j-1, j, j+1, j+2 \tag{1.36a}$$
$$j + j' \geq \kappa\ (= 2) \tag{1.36b}$$
$$m' = m-2, m-1, m, m+1, m+2. \tag{1.36c}$$

The supplementary conditions (1.35b,1.36b) are due to the triangle inequalities Eqs. (1.12).

The Wigner-Eckart theorem states that matrix elements can be divided into the product of a reduced matrix element that has no orientation dependence and an angular term that is independent of the norm of the operator. The reduced matrix element can be calculated most easily by considering particular values of m, m', q. For example

$$\langle n'j'||\hat{\mathbf{T}}_\kappa||nj\rangle = (-1)^{j'} \frac{\langle n'j'0|\hat{T}_{\kappa 0}|nj0\rangle}{\begin{pmatrix} j' & \kappa & j \\ 0 & 0 & 0 \end{pmatrix}}, \tag{1.37}$$

provided the denominator does not vanish.

To make these results more explicit let us consider an example. We wish to calculate the matrix element of a spherical harmonic Y_{1q} between orbital angular momentum states l, l'. We need the reduced matrix element which from Eq. (1.37) is[5]

$$\langle l'||\hat{Y}_1||l\rangle = (-1)^{l'} \frac{\langle l'0|\hat{Y}_{10}|l0\rangle}{\begin{pmatrix} l' & 1 & l \\ 0 & 0 & 0 \end{pmatrix}}.$$

We then use (1.30) to get

$$\langle l'0|\hat{Y}_{10}|l0\rangle = \sqrt{\frac{3}{4\pi}}\sqrt{(2l+1)(2l'+1)} \begin{pmatrix} l' & 1 & l \\ 0 & 0 & 0 \end{pmatrix}^2$$

and since

$$\begin{pmatrix} l' & 1 & l \\ 0 & 0 & 0 \end{pmatrix} = (-1)^{(l+l'+1)/2} \sqrt{\frac{l_{\max}}{(2l+1)(2l'+1)}}, \tag{1.38}$$

[5] We suppress the quantum number n in the braket since it labels a radial dependence and Y_{1q} has no dependence on r.

for l, l' integer and $l' = l \pm 1$, we have

$$\langle l'||\hat{Y}_1||l\rangle = (-1)^{l'+(l+l'+1)/2}\sqrt{\frac{3}{4\pi}}\sqrt{l_{\max}}.$$

Therefore

$$\begin{aligned}\langle l'm'|\hat{Y}_{1q}|lm\rangle &= (-1)^{l'-m'}\langle l'||\hat{Y}_1||l\rangle \begin{pmatrix} l' & 1 & l \\ -m' & q & m \end{pmatrix} \\ &= (-1)^{-m'+(l+l'+1)/2}\sqrt{\frac{3}{4\pi}}\sqrt{l_{\max}}\begin{pmatrix} l' & 1 & l \\ -m' & q & m \end{pmatrix} \qquad (1.39)\end{aligned}$$

which agrees with (1.30)[6].

We can use this result to calculate matrix elements of the position operator $\hat{\mathbf{r}}$. From (1.22) we have $\hat{\mathbf{r}} = r\sqrt{\frac{4\pi}{3}}\sum_q Y_{1q}\mathbf{e}_q^*$ so that

$$\langle n'l'm'|\hat{\mathbf{r}}|nlm\rangle = R_{n'l';nl}\sqrt{l_{\max}}\sum_q (-1)^{l'-2m'+(l+l'+1)/2}\begin{pmatrix} l' & 1 & l \\ -m' & q & m \end{pmatrix}\mathbf{e}_q^*.$$

The terms in the summation over q are nonzero only when $l' = l \pm 1$ and $m' = q + m$. We have introduced above the radial integral

$$R_{n'l';nl} = \int_0^\infty dr\ r^3 R_{n'l'}^* R_{nl}.$$

For hydrogenic atoms there is a closed form expression for $R_{n'l';nl}$ (The result is due to W. Gordon (1929) and is given in Bethe & Salpeter). For nonhydrogenic atoms the integral can be calculated numerically.

An important special case of the Wigner-Eckart theorem can be used to calculate matrix elements of vector operators without the need to evaluate reduced matrix elements. Consider a vector operator $\hat{\mathbf{A}}$. The Wigner-Eckart theorem implies that $\langle\alpha jm|\hat{\mathbf{A}}|\alpha jm'\rangle = C\langle\alpha jm|\hat{\mathbf{J}}|\alpha jm'\rangle$ where C is a constant and $\hat{\mathbf{J}}$ is the angular momentum operator. To evaluate the constant use

$$\begin{aligned}\langle\alpha jm|\hat{\mathbf{A}}\cdot\hat{\mathbf{J}}|\alpha jm\rangle &= \sum_{m'}\langle\alpha jm|\hat{\mathbf{A}}|\alpha jm'\rangle\cdot\langle\alpha jm'|\hat{\mathbf{J}}|\alpha jm\rangle \\ &= C\sum_{m'}\langle\alpha jm|\hat{\mathbf{J}}|\alpha jm'\rangle\cdot\langle\alpha jm'|\hat{\mathbf{J}}|\alpha jm\rangle \\ &= C\langle\alpha jm|\hat{\mathbf{J}}^2|\alpha jm\rangle \\ &= Cj(j+1)\hbar^2. \qquad (1.40)\end{aligned}$$

Thus

$$\langle\alpha jm|\hat{\mathbf{A}}|\alpha jm'\rangle = \frac{\langle\alpha jm|\hat{\mathbf{A}}\cdot\hat{\mathbf{J}}|\alpha jm\rangle}{j(j+1)\hbar^2}\langle\alpha jm|\hat{\mathbf{J}}|\alpha jm'\rangle. \qquad (1.41)$$

[6]Using Eq. (1.38) we get a phase factor $(-1)^{m'+(l+l'+1)/2}$ in (1.30) whereas (1.39) has a phase factor$(-1)^{-m'+(l+l'+1)/2}$. For $l' = l \pm 1$ and m' integer the phases are the same.

The result (1.41) is often referred to as the Landé projection theorem. Since the matrix elements of vector operators in a given subspace corresponding to a particular value of j are all proportional to each other the matrix elements of an operator $\hat{\mathbf{A}}$ can be calculated by projecting $\hat{\mathbf{A}}$ onto $\hat{\mathbf{J}}$, multiplying by the matrix elements of $\hat{\mathbf{J}}$, and normalizing by the expectation value of $\hat{\mathbf{J}}^2$ which is $\langle\hat{\mathbf{J}}^2\rangle = j(j+1)\hbar^2$.

1.5.2 Matrix elements of coupled angular momenta

In multielectron problems as well as when dealing with fine structure and hyperfine structure manifolds it is necessary to calculate matrix elements between states of coupled angular momenta.

When $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{s}}$ coupled states $|nls;jm\rangle$ are linear combinations of states in the uncoupled basis $|nlm_l;sm_s\rangle$. Applying the Wigner-Eckart theorem we have

$$\langle n'l's'j'm'|\hat{T}_{\kappa q}|nlsjm\rangle = (-1)^{j'-m'}\langle n'l's'j'||\hat{\mathbf{T}}_\kappa||nlsj\rangle \begin{pmatrix} j' & \kappa & j \\ -m' & q & m \end{pmatrix}. \tag{1.42}$$

When the tensor operator $\hat{\mathbf{T}}_\kappa$ commutes with $\hat{\mathbf{s}}$, the matrix element is only nonzero when $s = s'$ and we can use the result

$$\langle n'l'sj'||\hat{\mathbf{T}}_\kappa||nlsj\rangle = (-1)^{l'+s+j+\kappa}\langle n'l'||\hat{\mathbf{T}}_\kappa||nl\rangle\sqrt{(2j+1)(2j'+1)} \begin{Bmatrix} l' & j' & s \\ j & l & \kappa \end{Bmatrix}. \tag{1.43}$$

Here the symbol in curly braces is a 6j symbol which describes the coupling of three angular momenta. This result is true for any uncoupled angular momenta, i.e. we can replace $\hat{\mathbf{L}}$ by $\hat{\mathbf{J}}_1$ and $\hat{\mathbf{s}}$ by $\hat{\mathbf{J}}_2$ in the above expression.

The 6j symbol can be written as a sum over 3j symbols

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = \frac{(-1)^{j_1+j_2+j_4+j_5}}{\sqrt{(2j_3+1)(2j_6+1)}} \sum C^{j_5 m_5}_{j_3 m_3 j_4 m_4} C^{j_3 m_3}_{j_1 m_1 j_2 m_2} C^{j_5 m_5}_{j_1 m_1 j_6 m_6} C^{j_6 m_6}_{j_2 m_2 j_4 m_4}.$$

The sum is to be taken over all possible values of m_1, m_2, m_3, m_4, m_6 while m_5 is held fixed. In practice it is most convenient to look up values needed in a table.

1.6 Values of Clebsch-Gordan coefficients and 3j symbols

There exist a number of explicit expressions for the Clebsch-Gordan coefficents. An expression due to Wigner is

$$\begin{aligned} C^{c\gamma}_{a\alpha b\beta} &= \delta_{\gamma,\alpha+\beta}\Delta(abc)\left[\frac{(c+\gamma)!(c-\gamma)!(2c+1)}{(a+\alpha)!(a-\alpha)!(b+\beta)!(b-\beta)!}\right]^{1/2} \\ &\times \sum_z \frac{(-1)^{b+\beta+z}(c+b+\alpha-z)!(a-\alpha+z)!}{z!(c-a+b-z)!(c+\gamma-z)!(a-b-\gamma+z)!} \end{aligned} \tag{1.44}$$

where

$$\Delta(abc) = \sqrt{\frac{(a+b-c)!(a-b+c)!(-a+b+c)!}{(a+b+c+1)!}}$$

and in the summation z assumes all integer values for which the factorial arguments are nonnegative. The 3j symbols are then given by

$$\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} = (-1)^{2a+c+\gamma} \frac{1}{\sqrt{2c+1}} C^{c\gamma}_{a-\alpha b-\beta}.$$

For convenience I list some explicit formulae for 3j symbols

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix}.$$

The symbol is zero unless $m_1 + m_2 + m = 0$ and $|j_1 - j_2| \leq j \leq j_1 + j_2$.

Symmetry relations,

$$\begin{aligned}
\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} &= \begin{pmatrix} j_2 & j & j_1 \\ m_2 & m & m_1 \end{pmatrix} = \begin{pmatrix} j & j_1 & j_2 \\ m & m_1 & m_2 \end{pmatrix} \\
&= (-1)^{j_1+j_2+j} \begin{pmatrix} j_2 & j_1 & j \\ m_2 & m_1 & m \end{pmatrix} = (-1)^{j_1+j_2+j} \begin{pmatrix} j_1 & j & j_2 \\ m_1 & m & m_2 \end{pmatrix} \\
&= (-1)^{j_1+j_2+j} \begin{pmatrix} j & j_2 & j_1 \\ m & m_2 & m_1 \end{pmatrix}.
\end{aligned}$$

In addition,

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} = (-1)^{j_1+j_2+j} \begin{pmatrix} j_1 & j_2 & j \\ -m_1 & -m_2 & -m \end{pmatrix}. \tag{1.45}$$

Orthogonality relations,

$$\begin{aligned}
\sum_{jm} (2j+1) \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j \\ m_1' & m_2' & m \end{pmatrix} &= \delta_{m_1 m_1'} \delta_{m_2 m_2'} \\
\sum_{m_1 m_2} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j' \\ m_1 & m_2 & m' \end{pmatrix} &= \frac{\delta_{jj'} \delta_{m_1 m_1'}}{2j+1}.
\end{aligned} \tag{1.46}$$

For $j = j_1 + j_2$

$$\begin{aligned}
\begin{pmatrix} j_1 & j_2 & j_1 + j_2 \\ m_1 & m_2 & -m_1 - m_2 \end{pmatrix} &= (-1)^{j_1 - j_2 + m_1 + m_2} \\
&\times \sqrt{\frac{(2j_1)!(2j_2)!(j_1+j_2+m_1+m_2)!(j_1+j_2-m_1-m_2)!}{(2j_1+2j_2+1)!(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!}}.
\end{aligned} \tag{1.47}$$

For $m_1 = j_1$

$$\begin{pmatrix} j_1 & j_2 & j \\ j_1 & -j_1-m & m \end{pmatrix} = (-1)^{-j_1+j_2+m} \times\sqrt{\frac{(2j_1)!(-j_1+j_2+j)!(j_1+j_2+m)!(j-m)!}{(j_1+j_2+j+1)!(j_1-j_2+j)!(j_1+j_2-j)!(-j_1+j_2-m)!(j+m)!}}. \tag{1.48}$$

For $m_1 = m_2 = m = 0$. When $j_1 + j_2 + j = 2g$ with g an integer then

$$\begin{pmatrix} j_1 & j_2 & j \\ 0 & 0 & 0 \end{pmatrix} = (-1)^g \sqrt{\frac{(2g-2j_1)!(2g-2j_2)!(2g-2j)!}{(2g+1)!}} \frac{g!}{(g-j_1)!(g-j_2)!(g-j)!}.$$

For $m_1 = m_2 = m = 0$. When $j_1 + j_2 + j = 2g + 1$ with g an integer then

$$\begin{pmatrix} j_1 & j_2 & j \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

From these formulae we can write down specific results for $j = 0, 1/2, 1, 3/2...$.

For $j = 0$,

$$\begin{pmatrix} j_1 & j_2 & 0 \\ m_1 & m_2 & 0 \end{pmatrix} = (-1)^{-j_2-m_2} \frac{\delta_{j_1 j_2}\delta_{m_1,-m_2}}{\sqrt{2j_1+1}}.$$

For $j = 1/2$,

$$\begin{pmatrix} j+1/2 & j & 1/2 \\ m & -m-1/2 & 1/2 \end{pmatrix} = (-1)^{j-m-1/2} \sqrt{\frac{j-m+1/2}{(2j+1)(2j+2)}}.$$

For $j = 1$,

$$\begin{aligned}
\begin{pmatrix} j+1 & j & 1 \\ m & -m-1 & 1 \end{pmatrix} &= (-1)^{-j-m-1} \sqrt{\frac{(j-m)(j-m+1)}{(2j+1)(2j+2)(2j+3)}} \\
\begin{pmatrix} j+1 & j & 1 \\ m & -m & 0 \end{pmatrix} &= (-1)^{j-m-1} \sqrt{\frac{(j+m+1)(j-m+1)}{(2j+1)(j+1)(2j+3)}} \\
\begin{pmatrix} j & j & 1 \\ m & -m-1 & 1 \end{pmatrix} &= (-1)^{j-m} \sqrt{\frac{(j-m)(j+m+1)}{(j+1)(2j+1)2j}}. \\
\begin{pmatrix} j & j & 1 \\ m & -m & 0 \end{pmatrix} &= (-1)^{j-m} \frac{m}{\sqrt{(j+1)(2j+1)j}}.
\end{aligned} \tag{1.49}$$