

Error Correction in Ensemble Registers for Quantum Repeaters and Quantum Computers

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We propose to use a collective excitation blockade mechanism to identify errors that occur due to disturbances of single atoms in ensemble quantum registers where qubits are stored in the collective population of different internal atomic states. A simple error correction procedure and a simple decoherence-free encoding of ensemble qubits in the hyperfine states of alkali-metal atoms are presented.

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The potential of quantum computing, e.g., for factoring and unstructured search [1], lies in the significant speedup in comparison with classical computing allowed by the use of superposition states. Scalability of quantum devices to operations on a significant number of bits is crucial for all quantum computer proposals and is in principle achieved when information is stored in binary form in physically distinct two-level quantum systems since any unitary transformation on the full register product space can then be generated by a suitable set of single- and two-qubit operations. In atomic quantum computing proposals, for instance [2–4], bits are encoded in different atoms or ions, the collection of which forms a quantum register. In this case, scalability is not limited by any shortage of atoms in laboratory experiments, but by the immense difficulty of preserving and manipulating the quantum state of such a multicomponent system.

As an alternative to storage and encoding of quantum information in single quantum systems, it has been proposed to use atomic ensembles, which have favorably enhanced interactions with optical fields [5] and long coherence times despite the fact that even a single qubit is stored in a very entangled state of the many-atom system [6]. Encoding and storage of qubits in ensembles is a promising approach to repeater technologies for long-distance quantum communication [5,7], and recently we proposed an approach towards scalability of quantum computing which combines the ensemble ideas and the rich internal level structure found in many quantum systems [8]. Rather than storing individual bits in individual particles, we suggest to encode quantum information in the symmetric Fock space describing the collective population of the different internal states of the particles in the ensemble. One- and two-bit quantum gates as needed for quantum computing and for entanglement swapping and purification of a quantum repeater station based on the atomic multilevel structure may now be carried out by operations within the space of symmetric collective states of a single atomic ensemble.

An experimental difficulty consists of restricting the occupation of the internal states to the bit values zero and unity, as resonant driving of independent particles on an internal state transition naturally leads to multiple occupation of the same internal state. In atomic systems, however, one can force the system to remain in this desired subspace by use of the Rydberg blockade mechanism [3,9]. This phenomenon is due to the strong long-range interaction between highly excited atoms and the associated energy shifts, which prevent atoms in the vicinity of an excited neighbor from being transferred into a Rydberg state. Taking advantage of this mechanism one can drive precisely one atom from the initially macroscopically populated state $|s\rangle$, via a Rydberg state $|r\rangle$, into an initially empty internal state $|i\rangle$. One can also perform arbitrary single-qubit rotations [9], by (a) mapping $|i\rangle$ to $|r\rangle$ by a π pulse and (b) coupling the states with none and a single Rydberg excited atom for adjustable amounts of time, before (c) mapping the $|r\rangle$ amplitude back onto $|i\rangle$ with a final π pulse. Since the excitation to a Rydberg level from one of the internal states can prevent excitation from another internal state, two-qubit gates can be implemented through the same mechanism [8]. It is important to point out that the operations mentioned are carried out by collective addressing, and the exciting laser pulses only need to be adjusted to the energies and coupling strengths of the atomic states.

For our proposal to be practically useful, we now must address the errors which may occur and devise methods to repair them. Error correction techniques, derived under the assumption that errors occur independently on separate qubits, do not apply here since we work under the restriction that we can only collectively and symmetrically address the ensemble. Efficient error identification and correction is, however, possible in our framework, as shown below.

Let us first present an encoding scheme, which differs from the one proposed in [8], but which will greatly simplify our error analysis. We consider an ensemble of

K atoms with $2N + 1$ internal states, denoted $|s\rangle, |0_i\rangle, |1_i\rangle$, $i = 1, \dots, N$ (see Fig. 1). We assume that the system can be initially prepared in the reservoir product state $|S\rangle_K \equiv |ss\dots s\rangle$ and the atomic transitions $|0_i\rangle \leftrightarrow |1_i\rangle$, $i = 1, \dots, N$ can be driven selectively. Using the Rydberg blockade we produce the symmetric state of the ensemble with precisely one atom transferred from $|s\rangle$ into each of the $|0_i\rangle$ states, which encodes the logical state $|0_1, 0_2, \dots, 0_N\rangle$. In the following, we shall only apply Hamiltonians which leave the system in the symmetric Fock subspace corresponding to a single occupation of each manifold $\{|0_i\rangle, |1_i\rangle\}$, $i = 1, \dots, N$, at least until errors occur. One-bit rotations on qubit i are achieved by driving the single-atom transition $|0_i\rangle \leftrightarrow |1_i\rangle$ without involving excitation to Rydberg states and are thus faster and easier to implement than in [8,9]. Conditional two-bit dynamics still relies on long-range Rydberg interactions by using a π pulse, e.g., $|1_i\rangle \rightarrow |r\rangle$, to conditionally prevent another qubit $j \neq i$ from being excited into $|r\rangle$.

Because of interaction with their local environment (background gas collisions, spontaneous emission of radiation), it is physically motivated to assume that the individual particles which form a quantum register are affected by independent errors. In standard encoding schemes, where qubits are stored in separate particles, this assumption implies that qubits are corrupted independently of each other: error correction schemes then consist of syndrome measurement and conditional back action which corrects such errors if they are not too frequent [10]. In our ensemble encoding approach, the physical error model implies that the system may leave the symmetric, computational subspace. In the following paragraphs, we review the different types of errors which can affect the system and show how to correct them.

The first error source we shall consider is atom loss. For the sake of simplicity, we start with the case of a one-bit register, prepared in the logical 0 state, denoted $|\bar{0}\rangle_K$, to recall the symmetric collective character and the number K of atoms contributing to the ensemble state:

$$\begin{aligned} |\bar{0}\rangle_K &= \frac{1}{\sqrt{K}}(|0ss\dots s\rangle + |s0s\dots s\rangle + \dots + |ss\dots s0\rangle) \\ &= \frac{1}{\sqrt{K}}|0\rangle \otimes |S\rangle_{K-1} + \sqrt{\frac{K-1}{K}}|s\rangle \otimes |\bar{0}\rangle_{K-1}. \end{aligned} \quad (1)$$

The removal of any, say the first, atom, produces the state

$$|\psi\rangle = \frac{1}{\sqrt{K}}|S\rangle_{K-1} + \sqrt{\frac{K-1}{K}}|\bar{0}\rangle_{K-1} \quad (2)$$

or a mixed state with equivalent weight factors on the two components. For large K , this state is dominated by the second term, which encodes the correct register state but with a smaller total number of atoms in the ensemble. Since gates act on this state in the same way as on the state encoded with the original number of atoms, the error due to the atomic loss is therefore only connected with the first component in (2), which will be propagated by the sub-

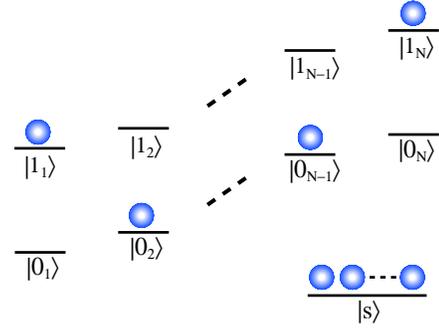


FIG. 1 (color online). Qubit encoding in the symmetric states of an ensemble of $(2N + 1)$ -level systems. $|s\rangle$ denotes the reservoir state. The figure depicts the state $|10\dots 01\rangle$.

sequent unitary dynamics and introduce a very small probability (i.e., the probability that the loss occurred multiplied with the factor $1/K$) for an erroneous output at the end of the calculation. The same reasoning can be applied to any qubit superposition state yielding the same result, and a simple calculation shows that in an N bit register, loss of a single atom introduces an erroneous component with population $\frac{N}{K}$. For very large K we may accept the erroneous component in the wave function, or we may identify the error, by carrying out a measurement to find out if one of the subspaces $\{|0_i\rangle, |1_i\rangle\}$ is not populated. By transferring an atom from $|s\rangle$ to that subspace via the Rydberg state, we reestablish a legal, but most likely erroneous, register state. The error can, however, be addressed by suitable error correction techniques, which we shall outline below.

Atom loss is not the most critical error to affect the ensemble since the resulting state retains the full permutation symmetry, and the computation may go on safely. It is more problematic when an atom is disturbed and remains in the sample where it continues to interact with the other atoms and corrupts the future quantum gates.

For the sake of simplicity, let us examine the case of a single-bit register initially prepared in the single-qubit state:

$$\begin{aligned} \alpha|\bar{0}\rangle_K + \beta|\bar{1}\rangle_K &= \frac{1}{\sqrt{K}}(\alpha|0\rangle + \beta|1\rangle) \otimes |S\rangle_{K-1} \\ &+ \sqrt{\frac{K-1}{K}}|s\rangle \otimes (\alpha|\bar{0}\rangle_{K-1} + \beta|\bar{1}\rangle_{K-1}). \end{aligned} \quad (3)$$

If the first atom is affected by an error, the term in the first line has negligible amplitude compared to the term in the second line if K is large, and it will be neglected in the following. Suppose that the $|s\rangle$ state of the first atom evolves into

$$|\phi\rangle = c_0|0\rangle + c_1|1\rangle + c_s|s\rangle + |\phi'\rangle, \quad (4)$$

where $|\phi'\rangle$ is orthogonal to $|0\rangle, |1\rangle$, and $|s\rangle$, and that the resulting erroneous state of the ensemble thus takes the form

$$\begin{aligned}
|\psi_{\text{er}}\rangle = & \sqrt{\frac{K-1}{K}}(c_s|s\rangle + |\phi'\rangle) \otimes (\alpha|\bar{0}\rangle_{K-1} + \beta|\bar{1}\rangle_{K-1}) \\
& + \sqrt{\frac{K-1}{K}}(c_0|0\rangle + c_1|1\rangle) \otimes (\alpha|\bar{0}\rangle_{K-1} + \beta|\bar{1}\rangle_{K-1}).
\end{aligned} \tag{5}$$

In the first line, the $|\phi'\rangle$ component will never interfere with the computation, and the single atom state $|s\rangle$ couples so weakly to our laser driving fields in comparison with the symmetric states of the remaining atoms that it will only slightly perturb the state of the ensemble. This robustness against perturbations on individual atoms is crucial for many uses of ensembles [6], and has been crucial in experiments, e.g., on continuous variable quantum storage in atomic ensembles, which already in their initial state may have a significant number of atoms that are not pumped into the symmetric state [11]. The second line in Eq. (5),

$$\sqrt{\frac{K-1}{2K}}[(c_0|r_0\rangle + c_1|r_1\rangle) \otimes (\alpha|\bar{0}\rangle_{K-1} + \beta|\bar{1}\rangle_{K-1}) + (c_0|0\rangle + c_1|1\rangle) \otimes (\alpha|\bar{r}_0\rangle_{K-1} + \beta|\bar{r}_1\rangle_{K-1})].$$

We can now check whether an error has occurred or not by measuring the Rydberg state content in the ensemble, e.g., by means of the blockade of a neighboring readout ensemble [13]. Such a measurement is projective and if a Rydberg excitation in $|r_0\rangle$ is detected we get the unnormalized state

$$c_0|r_0\rangle \otimes (\alpha|\bar{0}\rangle_{K-1} + \beta|\bar{1}\rangle_{K-1}) + \alpha(c_0|0\rangle + c_1|1\rangle) \otimes |\bar{r}_0\rangle_{K-1}.$$

The goal is now to modify this state so that it can be used for further processing. Since the coupling of the symmetrically excited Rydberg state $|\bar{r}_0\rangle_{K-1}$ to the reservoir state $|S\rangle_{K-1}$ is $\sqrt{K-1}$ times larger than the coupling of the single atom states $|r_0\rangle$ and $|s\rangle$, a resonant π pulse on the $s \leftrightarrow r_0$ transition can be driven, which makes a complete transfer of the collective state $|\bar{r}_0\rangle$ into $|S\rangle$ and only a $\sim 1/\sqrt{K}$ transfer of the single atom component $|r_0\rangle$ into $|s\rangle$ [14].

After such a pulse we therefore mainly obtain

$$c_0|r_0\rangle \otimes (\alpha|\bar{0}\rangle_{K-1} + \beta|\bar{1}\rangle_{K-1}) + \alpha(c_0|0\rangle + c_1|1\rangle) \otimes |S\rangle_{K-1}.$$

We then apply an ionizing pulse of the Rydberg state $|r_0\rangle$. If an ion is observed we retain the correct initial qubit state $(\alpha|\bar{0}\rangle_{K-1} + \beta|\bar{1}\rangle_{K-1})$ with one atom less in the ensemble. If no ion is detected, we obtain $(c_0|0\rangle + c_1|1\rangle) \otimes |S\rangle_{K-1}$. One then applies pulses ionizing the states $|0\rangle$ and $|1\rangle$, which finally leads to $|S\rangle_{K-1}$. We complete the procedure by applying a resonant pulse $s \leftrightarrow 0$ so that $|S\rangle_{K-1}$ is transferred into $|\bar{0}\rangle_{K-1}$.

If, during the first Rydberg state measurement, $|r_0\rangle$ is not observed, and if $|r_1\rangle$ is subsequently detected, one can apply similar procedures as above. If neither $|r_0\rangle$ nor $|r_1\rangle$ are detected, the system is projected onto a (sufficiently) nonerroneous component in Eq. (5) that the ensemble still reliably encodes the desired register state.

however, shows dangerous double occupancy of register states 0, 1 which must be suppressed by physical manipulation of the system.

To this end, we apply the following procedure, which is illustrated in Fig. 2. We first simultaneously apply two laser beams coupling $|0\rangle$ and $|1\rangle$ to two different Rydberg states $|r_0\rangle$ and $|r_1\rangle$ with the same Rabi frequency Ω . We assume that both Rydberg states $|r_0\rangle$ and $|r_1\rangle$ block multiple Rydberg excitations. As a consequence, the states $|x\rangle \otimes |\bar{y}\rangle_{K-1}$, with $x = s, \phi'$ and $y = 0, 1$, couple to the states $|x\rangle \otimes |\bar{r}_y\rangle_{K-1}$ with the same coupling strength $\hbar\Omega$, while the states with $x = 0, 1$ and $y = 0, 1$ couple to $(|r_x\rangle \otimes |\bar{y}\rangle_{K-1} + |x\rangle \otimes |\bar{r}_y\rangle_{K-1})/\sqrt{2}$, with the coupling strength $\sqrt{2}\hbar\Omega$. Thanks to the second coupling strength being larger by a factor of $\sqrt{2}$, it is possible to design a composite pulse sequence [12] which leaves the first line in Eq. (5) (and also any nonerroneous state) unchanged while transforming the second line into

By applying the detection or correction procedure described above to a single-qubit register, we either get the correct initial state $(\alpha|\bar{0}\rangle_{K-1} + \beta|\bar{1}\rangle_{K-1})$ or end up in $|\bar{0}\rangle_{K-1}$. It should be emphasized that as long as no error or a simple particle loss occurs, no error signal is reported by the Rydberg populations, and no atoms are unnecessarily removed from the ensemble. In an N -bit register one merely has to check all the qubit populations successively as described above. At the end, we know whether and where an error has occurred. If needed, we can repair the state of the erroneous qubit, provided we encode each bit of information in a superposition state $\alpha(|\bar{0}\bar{0}\rangle_K + |\bar{1}\bar{1}\rangle_K)/\sqrt{2} + \beta(|\bar{0}\bar{1}\rangle_K + |\bar{1}\bar{0}\rangle_K)/\sqrt{2}$, such that two ensemble qubits encode one logical qubit. Going through the above error identification and register restoration scheme, assuming an error has happened in any of the two register positions, we either recover the same superposition, but with one atom less in the ensemble, or superpositions involving only the first or the second component of our two-bit code words, such as $\alpha|\bar{0}\bar{0}\rangle_{K-1} + \beta|\bar{0}\bar{1}\rangle_{K-1}$, from which the entire code words can be reconstructed by simple gate operations. This error-correction encoding is simpler than in the usual tensor product encoding because our error identification protocol provides direct information about which qubit has to be repaired. We anticipate that other schemes may be derived to fix errors without ionization of the atoms, but the reduction in size of the symmetric ensemble due to errors seems unavoidable. If p denotes the error probability per atom per gate, the number of gates n_g must fulfill $pn_g \lesssim 1$ to maintain the correct computational state component against single atom loss, and $(pK)^2 n_g \lesssim 1$ to limit the number of simultaneous pairs of errors. A value of $p = 10^{-4}$ is thus compatible with the application of as much as 10^4 quantum gates on a few tens

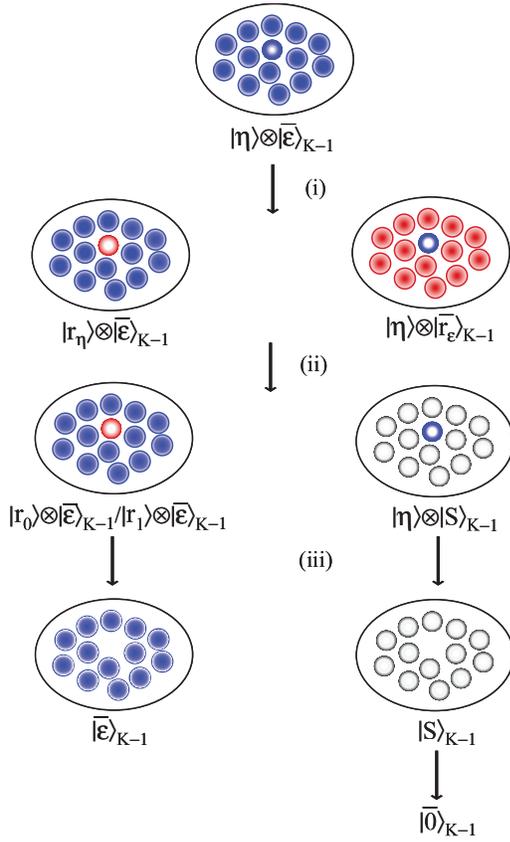


FIG. 2 (color). Schematic illustration of our procedure to remove single atom occupancy of register states, already occupied by the ensemble. The second and third lines of the figure depict superposition states. We use the shorthand notation $|\eta\rangle = c_0|0\rangle + c_1|1\rangle$, $|r_\eta\rangle = c_0|r_0\rangle + c_1|r_1\rangle$, $|\epsilon\rangle = \alpha|\bar{0}\rangle_{K-1} + \beta|\bar{1}\rangle_{K-1}$, and $|r_\epsilon\rangle = \alpha|\bar{r}_0\rangle_{K-1} + \beta|\bar{r}_1\rangle_{K-1}$. Our error detection sequence proceeds as follows: (i) $0 \leftrightarrow r_0$ and $1 \leftrightarrow r_1$ resonant composite pulses are applied. (ii) After detection of either $|r_0\rangle$ or $|r_1\rangle$, either a $s \leftrightarrow r_0$ or a $s \leftrightarrow r_1$ resonant pulse is applied. (iii) Depending on the result of the measurement in (ii) either $|r_0\rangle$ or $|r_1\rangle$ is ionized. If an ion signal is observed, the final state is $|\bar{\epsilon}\rangle_{K-1}$; otherwise $|\eta\rangle$ is ionized, yielding the state $|S\rangle_{K-1}$, subsequently transferred into the final state $|\bar{0}\rangle_{K-1}$ by a $s \leftrightarrow 0$ resonant pulse.

of qubits in a hundred-atom-large ensemble. We note that it is in principle possible to transfer the quantum state of our diminished ensemble by the Rydberg blockade mechanism to a nearby independent ensemble of atoms, and such new ensembles can be supplied when needed.

If the qubits are encoded in the internal Zeeman sub-levels of the lower atomic hyperfine levels in alkali-metal atoms with ground hyperfine manifolds $\{f, f+1\}$, a bias magnetic field along the z axis both makes the system immune to fluctuating orthogonal x and y components of the field and provides an energy splitting of the qubit states so that they can be unambiguously addressed by resonant laser fields [8]. Fluctuations of the z component of the field perturb the atomic energy levels, but we can make the system immune to these fluctuations if qubits are associated with the pairs of states $\{|f, m_f\rangle, |f+1, -m_f\rangle\}$, with

$m_f = 0, \pm 1, \pm 2, \dots, \pm f$. The states $|f, m_f\rangle, |f+1, -m_f\rangle$ experience the same linear Zeeman shift and only small differential quadratic shifts in the presence of magnetic field fluctuations, which therefore only slightly disturb the relative phase of qubit states 0 and 1. Coherence times exceeding 1 s have thus been observed for superpositions of such states with $m_f = \pm 1$ in ^{87}Rb [15].

In conclusion, we have proposed a novel scheme for encoding multiple bits of quantum information in symmetric collective states of an ensemble of multilevel atoms by making effective use of the single particle Hilbert space dimension. Errors on individual atoms can be repaired successfully if the ensemble is large enough. We furthermore showed that a specific choice of qubit levels provides an automatic decoherence-free subspace encoding against fluctuations in the external magnetic field acting on all atoms. We presented the analysis for the special case of a global Rydberg blockade mechanism, but we anticipate that the scheme presented here for identifying and eliminating systems which have left the symmetric subspace may apply also with other effective interactions as a general restoration mechanism for symmetric states in ensembles of identical particles involved in quantum computing, long-term quantum memories, and in quantum repeaters.

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