Self-Induced Dipole Force and Filamentation Instability of a Matter Wave

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The interaction of copropagating electromagnetic and matter waves is described with a set of coupled higher-order nonlinear Schrödinger equations. Optical self-focusing modulates an initially planar wave leading to the generation of dipole forces on the atoms. Atomic channeling due to the dipole forces leads, in the nonlinear regime, to filamentation of the atomic beam. Instability growth rates are calculated for atomic beams with both low and high phase space densities. In one transverse dimension an exact solution is found that describes a coupled optical and atomic soliton. [S0031-9007(98)06469-2]

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Tremendous advances in the manipulation of atomic motion by resonant and near-resonant interaction with optical fields have been made in recent years [1]. Atomic analogs of a number of well known optical devices and effects including interferometers [2,3], frequency shifters [4], fiber guiding [5], and lasers [6] have been demonstrated. Until now experimental attention has focused on the regime of *linear* atom-light interactions where the atomic dynamics are driven by externally specified optical fields. As has been discussed theoretically by several groups [7–9], dipole-dipole interactions lead to a light mediated nonlinearity in ultracold atomic beams where the thermal de Broglie wavelength is large compared to the average interatomic spacing. The resulting nonlinear Schrödinger equation for the atomic evolution suggests the possibility of observing nonlinear effects such as atomic self-focusing and solitons, in copropagating matter and optical waves [7,10].

In this Letter coupled equations are formulated that describe the joint evolution of copropagating matter and optical waves. Including the effects of diffraction, and nonlinearity, the evolution of both the optical and the atomic fields leads to new physical phenomena. The dipole-dipole nonlinearity is only significant when the matter wave has long range spatial coherence, such that the dimensionless phase space density $\rho_{\rm ps} = \mathcal{N}\lambda_{\rm dB}^3$ is order unity, where $\mathcal{N}$ is the atomic density and $\lambda_{\rm dB}$ is the de Broglie wavelength. However, even in the limit of short coherence length where $\rho_{\rm ps} \ll 1$ the single atom dipole nonlinearity can lead to strong perturbations of the atomic motion. The effect of the dipole potential was discussed by Askar'yan many years ago [11], and later demonstrated experimentally [12]. Dipole forces, being proportional to $\nabla|A|^2$, where A is the envelope of the optical field, occur only in spatially inhomogeneous fields. However, spatial gradients grow on an initially homogeneous background due to modulational instability under conditions of nonlinear self-focusing of the optical field [13]. The spatial gradients result in dipole forces so that the atomic motion may be subject to forces nonlinear in the atomic density, even in the case of an atomic beam with low phase space density. The nonlinear stage of the instability of copropagating optical and matter waves leads to filamentation of the atomic beam. The coupled evolution equations also predict the possibility of self-trapped propagation, for particular initial conditions, of both the optical and the atomic fields.

Consider a matter wave copropagating with an electromagnetic wave along the z axis. In the limit of large detuning between the light frequency and the atomic transition the upper level can be eliminated [7], giving a scalar Schrödinger equation for the evolution of the atomic ground state: $i\hbar\partial_t\Psi = (-\frac{\hbar^2}{2m}\nabla^2 + V)\Psi$, where m is the atomic mass and $V = V(\vec{r})$ is the position dependent potential. The rapid phase dependence associated with the mean velocity v_a is removed by putting $\Psi = \psi e^{i(k_a z - \Omega_0 t)}$, where $k_a = mv_a/\hbar$, $\Omega_0 = mv_a^2/(2\hbar)$, to obtain the paraxial Schrödinger equation

$$i\hbar(\partial_t + v_a\partial_z)\psi = \left(-\frac{\hbar^2}{2m}\nabla_\perp^2 + V\right)\psi\,, \qquad (1)$$

where we have used $\partial_{zz}\psi \ll \nabla_\perp^2\psi$, with $\nabla_\perp^2 \equiv \partial_{xx} + \partial_{yy}$.

The potential may be written in the form $V = V_{\rm d} + V_{\rm dd} + V_{\rm coll}$. The single atom dipole potential is given by [14] $V_{\rm d} = \frac{\hbar\delta}{2}\ln(1 + \frac{1}{1+4\delta^2/\gamma^2}\frac{|A|^2}{I_s})$, where the detuning is $\delta = \omega - \omega_a - kv_a$, ω is the optical frequency, ω_a is the atomic transition frequency, $k = 2\pi/\lambda = \omega/c$, λ is the optical wavelength, c is the speed of light, γ is the homogeneous linewidth, and I_s is the saturation intensity.

When treating an atomic beam with long range coherence we must add the contribution due to light mediated dipole-dipole interactions. The dipole-dipole interaction is nonlocal and involves an integral over relative atomic coordinates [7,8]. In the limit of $\lambda_{\rm dB} \gg \lambda$ the dipole-dipole interaction can be approximated by a contact potential of the form $V_{\rm dd} = -V_{\rm d}V_0\frac{\gamma}{\delta}|\psi|^2$, where V_0 arises from an overlap integral and has the dimensions of a volume [7]. The overlap integral is dominated by contributions within one coherence length so that $V_0 = V_0(T) \approx \lambda_{\rm dB}^3$, where $\lambda_{\rm dB} = \sqrt{2\pi\hbar^2/mk_BT}$. Note that irrespective of the sign of the detuning $V_{\rm dd} < 0$, so the dipole-dipole interaction acts as an attractive nonlinearity for the atomic dynamics.

The nonlinear term due to two body collisions is $V_{\rm coll} = Q|\psi|^2$, where $Q = 4\pi\hbar^2 a/m$, with a the scattering length (see, for example, [15]). For positive (negative) scattering lengths the interaction is repulsive (attractive). Phenomenologically we may consider Q as a continuous parameter with $Q = 0$ for $\rho_{\rm ps} \ll 1$, and $Q = \pm Q_{\rm max}$ in the zero temperature limit.

Time independent propagation of the optical field $E = A\exp[i(kz - \omega t)] +$ c.c. is described in the paraxial approximation by a parabolic equation for the slowly varying envelope A of the form [16]

$$\partial_z A - \frac{i}{2k}\nabla_\perp^2 A = -i\frac{6\pi}{k^2} \times \left(\frac{\delta/\gamma}{1 + 4\delta^2/\gamma^2 + |A|^2/I_s}\right)\mathcal{N}A\,. \quad (2)$$

Consistent with the large detuning approximation we have neglected linear and nonlinear absorption in Eq. (2). The appearance of the atomic density $\mathcal{N}$ on the right-hand side of Eq. (2) couples the electromagnetic propagation to the atomic dynamics. Normalizing the wave function in terms of the total particle number $N_T = \int d^3\vec{r}|\psi|^2$ we have $\mathcal{N} = |\psi|^2$ [17].

Combining Eqs. (1) and (2), assuming weak saturation $|A|^2 \ll I_s$, and restricting attention to time independent propagation give the coupled evolution equations

$$\partial_z\psi - i\alpha_1\nabla_\perp^2\psi = -i[(s_1 - \beta_1|\psi|^2)|A|^2 + s_2\beta_2|\psi|^2]\psi\,, \quad (3a)$$

$$\partial_z A - i\nabla_\perp^2 A = -is_1(1 - |A|^2)|\psi|^2 A\,. \quad (3b)$$

The dimensionless parameters are $\alpha_1 = k/k_a$, $\beta_1 = k^2\gamma V_0/(12\pi v_a\mathcal{L})$, $\beta_2 = k^2|Q|/(6\pi\hbar v_a\mathcal{L})$, $s_1 = \mathrm{sgn}(\delta)$, and $s_2 = \mathrm{sgn}(Q)$, where $\mathcal{L} = (|\delta|/\gamma)/(1 + 4\delta^2/\gamma^2)$. The fields and coordinates have been normalized as $|\psi|^2/[k^2|\delta|/(12\pi v_a\mathcal{L})] \to |\psi|^2$, $|A|^2/[|\delta|I_s/(\gamma\mathcal{L})] \to |A|^2$, $z/(2v_a/|\delta|) \to z$, and $(x,y)/\sqrt{v_a/(k|\delta|)} \to (x,y)$.

Equations (3) describe the coupled nonlinear evolution of matter and optical fields. They differ from previous work on nonlinear propagation of atomic waves in that the nonlinear and diffractive terms are included for the matter *and* the optical fields. The result is a consistent description of the nonlinear coupling of the two fields that leads to new physics in the form of self-induced dipole forces.

Before proceeding to analysis of Eqs. (3) it is of interest to estimate values for the parameters that appear. The factor α_1 describes the relative strength of atomic and optical diffraction, and is inversely proportional to the velocity v_a. α_1 is unity when the atomic velocity is equal to the single photon recoil velocity: $v_a = v_r = \hbar k/m$. The variation of the dimensionless nonlinear terms is shown in Fig. 1 as a function of the normalized detuning δ/γ. Figure 1a shows the behavior typical of a low phase space density beam ($\rho_{\rm ps} \sim 10^{-6}$), while Fig. 1b corresponds to $\rho_{\rm ps} \sim 3.7$. The nonlinear terms $V_{\rm d}$, $V_{\rm dd}$, and $V_{\rm coll}$ scale, after normalization, as δ^{-2}, δ^{-3}, and δ^{-1}, respectively. Thus the collisional term always dominates at sufficiently large detunings. An obvious approach to experimental generation of suitable copropagating optical and atomic beams would be to use the radiation pressure output coupler demonstrated recently [18]. In that work the experimental parameters were not far from the example given in Fig. 1a.

To analyze the modulational instability of Eqs. (3) we start by noting they admit stationary plane-wave solutions $\bar{\psi} = \psi_0 e^{i\theta_1 z}$, $\bar{A} = A_0 e^{i\theta_2 z}$ with $\theta_1 = -(s_1 - \beta_1\psi_0^2)A_0^2 - s_2\beta_2\psi_0^2$, and $\theta_2 = -s_1(1 - A_0^2)\psi_0^2$. We then make the ansatz

$$\psi = \bar{\psi}[1 + (\delta\psi_+ e^{iqx} + \delta\psi_-^* e^{-iqx})e^{sz}]\,, \quad (4)$$

$$A = \bar{A}[1 + (\delta A_+ e^{iqx} + \delta A_-^* e^{-iqx})e^{sz}]\,, \quad (5)$$

linearize Eqs. (3), and keep terms that are transversely phase matched to obtain the eigenvalue equation $M\vec{a} = is\vec{a}$ for the growth rate s where $\vec{a} = (\delta\psi_+, \delta\psi_-, \delta A_+, \delta A_-)$ and

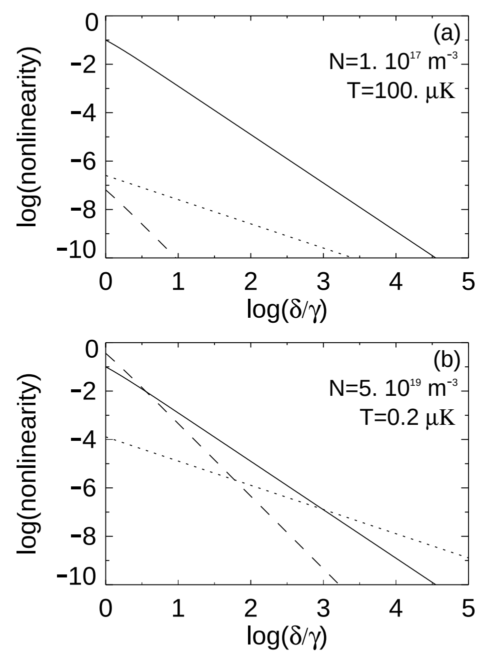

FIG. 1. Nonlinear potentials for (a) low and (b) high phase space density as a function of detuning. The curves depict the normalized nonlinearities $V_{\rm d} = |A|^2$ (solid line), $V_{\rm dd} = \beta_1|\psi|^2|A|^2$ (dashed line), and $V_{\rm coll} = \beta_2|\psi|^2$ (dotted line). Numerical parameters for ^{87}Rb were used, and $|A|^2/I_{\rm sat} = 0.5$.

$$M = \begin{pmatrix} -\beta_1\psi_0^2A_0^2 + s_2\beta_2\psi_o^2 + \alpha_1 q^2 & -\beta_1\psi_0^2A_0^2 + s_2\beta_2\psi_0^2 & s_1A_0^2 - \beta_1\psi_0^2A_0^2 & s_1A_0^2 - \beta_1\psi_0^2A_0^2 \\ \beta_1\psi_0^2A_0^2 - s_2\beta_2\psi_0^2 & \beta_1\psi_0^2A_0^2 - s_2\beta_2\psi_o^2 - \alpha_1 q^2 & -s_1A_0^2 + \beta_1\psi_0^2A_0^2 & -s_1A_0^2 + \beta_1\psi_0^2A_0^2 \\ s_1\psi_0^2 - s_1A_0^2\psi_0^2 & s_1\psi_0^2 - s_1A_0^2\psi_0^2 & -s_1A_0^2\psi_0^2 + q^2 & -s_1A_0^2\psi_0^2 \\ -s_1\psi_0^2 + s_1A_0^2\psi_0^2 & -s_1\psi_0^2 + s_1A_0^2\psi_0^2 & s_1A_0^2\psi_0^2 & s_1A_0^2\psi_0^2 - q^2 \end{pmatrix}. \tag{6}$$

The instability growth rates $[\mathrm{Re}(s)]$ found from solving the resulting quadratic equation for s^2 are shown in Fig. 2 as a function of the transverse wave number q. Note that, in contrast to the instability of an optical beam in self-focusing media that requires $s_1 = 1$, modulational instability of Eqs. (3) occurs for all combinations of s_1, s_2. Observability of a convective instability is dependent on both the growth rate and the initial level of noise. We may estimate the necessary interaction length by requiring, for example, that density variations are amplified by a factor of 10^3 at the most unstable spatial scale. The corresponding length turns out to be a modest 1 cm, and 0.2 mm, for conditions corresponding to Figs. 2a and 2b, respectively.

The nonlinear stage of the instability leads to filamentation, as can be seen from the numerical solutions of Eqs. (3), shown in Fig. 3 where high intensities and densities are colored white. For the parameters used in Fig. 3 the atomic dynamics are self-defocusing, whereas the highest order term in the equation for the optical field is self-focusing. This results in filamentation patterns that are anticorrelated in the two fields, as can be seen in the figure.

In addition to the previous modulational instability that occurs for plane-wave inputs Eqs. (3) admit localized solutions that may be attractors for appropriate initial conditions. We note here that an exact one-dimensional solitary solution for the coupled optical and atomic propagation exists for $s_1 = 1$, and $\alpha_1 = \beta_1 = 1$, $\beta_2 = 0$. Under these conditions we may put $\psi = A = U$, and seek a self-trapped solution in the form $U = u(x)e^{i\theta z}$, where

$$u_{xx} = \theta u + u^3 - u^5. \tag{7}$$

Equation (7) has the solution [19]

$$u^2 = \frac{4a\theta}{4\theta\cosh^2(\sqrt{\theta}\,x) - a - 2\theta}, \tag{8}$$

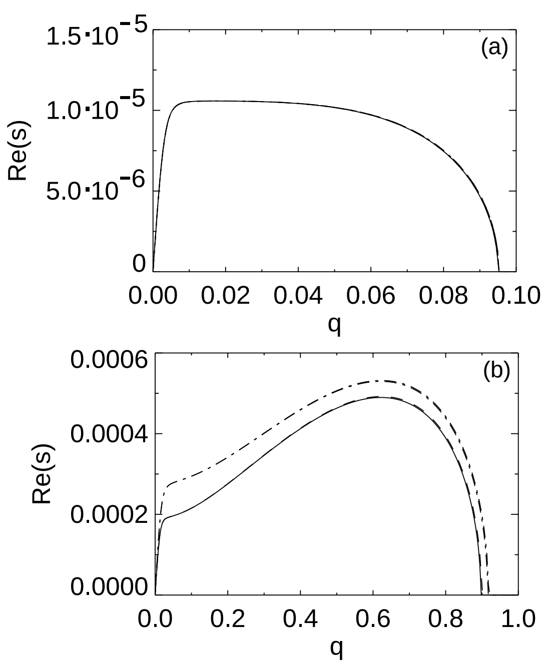

FIG. 2. Instability growth rates for (a) low and (b) high phase space densities. In frame (a) the four curves corresponding to $(s_1, s_2) = (\pm 1, \pm 1)$ are all plotted, but are virtually indistinguishable. In frame (b) the upper curves correspond to $(s_1, s_2) = (-1, \pm 1)$, while the lower curves correspond to $(s_1, s_2) = (+1, \pm 1)$. Parameters were the same as those used in the corresponding frames in Fig. 1, with $\delta/\gamma = 10$ and $v_a = 5$ m/s which gives $\alpha_1 \sim 0.001$.

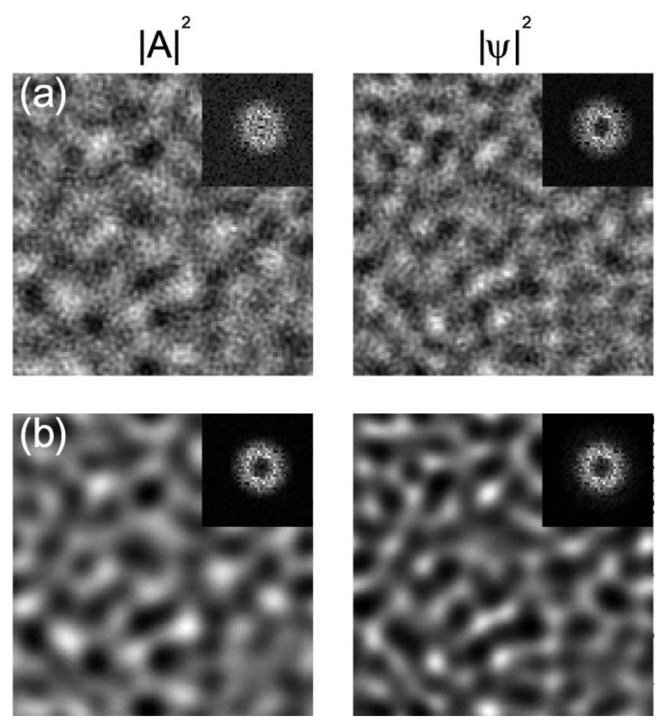

FIG. 3. Filamentation patterns at (a) $z = 55$ and (b) $z = 110$, for $s_1 = 1, \alpha_1 = 0.1, \beta_1 = 0, \beta_2 = 0$, and $|A_0|^2 = |\psi_0|^2 = 0.1$. The insets show the spatial power spectrum. The width of the computational region was $l_x = l_y = 100$, and the computation was initiated by adding fluctuations of mean level 10^{-5} to both fields.

where $a = [12\theta^2/(3 + 16\theta)]^{1/2}$. The propagation constant is related to the number of particles $N_T = \int_{-\infty}^{\infty} dx\, u^2$ by $\theta = (3/16)\tan^2(N_T/\sqrt{3})$. The criterion $dN_T/d\theta > 0$ [20] is thus satisfied which is consistent with stability of the self-trapped solution. Note, however, that since Eq. (7) was obtained by reducing the coupled Eqs. (3) with the ansatz $\psi = A$ the criterion [20] does not guarantee stability. Future work will deal with solitary solutions of the set (3), and their stability, also for $\psi \neq A$.

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Note added.— The possibility of coupled self-focusing of atomic and light beams was also considered in Ref. [21].

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